Analysis of Port-Hamiltonian Systems

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1 Introduction
1.1 Recapitulate

In the first parts we have seen that

- (Hyperbolic) pde’s lead to Stokes-Dirac structures,
- Stokes-Dirac structure gives pde’s. $f = \dot{x}(t)$ and $e = \dfrac{\delta H}{\delta x}$
Example

Define $\mathcal{F} = \mathcal{E} = C^\infty([a, b])$, and consider the Stokes-Dirac structure

$$\mathcal{D} = \left\{ \left( \begin{array}{c} f \\ f_\delta \end{array} \right), \left( \begin{array}{c} e \\ e_\delta \end{array} \right) \in \mathcal{F} \times \mathcal{E} \mid f = \frac{\partial e}{\partial \zeta} \right\} \text{ and }$$

$$f_\zeta = \frac{1}{\sqrt{2}}(e(a) - e(b)), \quad e_\zeta = \frac{1}{\sqrt{2}}(e(a) + e(b))$$

This is a Stokes-Dirac structure on $\mathcal{F} \times \mathcal{E}$. 
Now we define \( f = \dot{x}(t) \) and \( e = x \).

Thus \( e = \frac{\delta H}{\delta x} \) for \( H(x) = \frac{1}{2}x^2 \).

With these choices we obtain the pde on \( \zeta \in [a, b] \)

\[
\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t)
\]

To have a solution, we need a boundary condition.

For \( f_\partial = 0 \), i.e., \( x(b, t) = x(a, t) \) there is in general no solution which stays in \( \mathcal{E} \). Although the solution would satisfy \( H(x(t)) \) is constant (as function of \( t \)).
Hence Stokes-Dirac structure do not give automatically existence of a solution for a corresponding pde.

However, they do give properties of these solutions, provided they exist.
1.2 Our class of pde’s

Example

\[
\begin{align*}
V(a) & \quad I(a) \\
V(b) & \quad I(b)
\end{align*}
\]

The pde model of the transmission line is given by:

\[
\begin{align*}
\frac{\partial Q}{\partial t}(\zeta, t) &= - \frac{\partial}{\partial \zeta} \frac{\phi(\zeta, t)}{L(\zeta)} \\
\frac{\partial \phi}{\partial t}(\zeta, t) &= - \frac{\partial}{\partial \zeta} \frac{Q(\zeta, t)}{C(\zeta)}.
\end{align*}
\]
Introducing $x_1 = Q$ and $x_2 = \phi$, this pde becomes

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (\zeta, t) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta} \begin{pmatrix} \frac{1}{C(\zeta)} x_1(\zeta, t) \\ \frac{1}{L(\zeta)} x_2(\zeta, t) \end{pmatrix}$$

(2)

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta} \left[ \begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix} \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix} \right]$$

$$= P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x) (\zeta, t).$$
So we have that the transmission line can be written as

\[
\frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} (\zeta, t) = P_1 \frac{\partial}{\partial \zeta} \left( \mathcal{H} x \right) (\zeta, t),
\]

with \( P_1 \) and \( \mathcal{H} \) given by

\[
P_1 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix}.
\]

Note that \( \mathcal{H} \) is non-uniform.
Furthermore, the energy associated to the transmission line,

\[
E(t) = \frac{1}{2} \int_a^b \frac{Q(\zeta, t)^2}{C(\zeta)} + \frac{\phi(\zeta, t)^2}{L(\zeta)} d\zeta
\]

can be written as

\[
E(t) = \frac{1}{2} \int_a^b \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix}^T \begin{pmatrix} \frac{1}{C(\zeta)} & 0 \\ 0 & \frac{1}{L(\zeta)} \end{pmatrix} \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix} d\zeta
\]

\[
= \frac{1}{2} \int_a^b \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix}^T \mathcal{H}(\zeta) \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix} d\zeta.
\] (5)
Summarizing, we have

\[
\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial}{\partial t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} (\mathcal{H} x)(\zeta, t)
\]

and

\[
E(t) = \frac{1}{2} \int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta.
\]

We have seen that \( P_1 \) and \( \mathcal{H}(\zeta) \) are real symmetric matrices.

This is enough for obtaining the power balance.
\[
\frac{dE}{dt}(t) = \frac{1}{2} \int_a^b \frac{\partial x}{\partial t}(\zeta, t)^T \mathcal{H}(\zeta)x(\zeta, t)\,dz + \\
\frac{1}{2} \int_a^b x(\zeta, t)^T \mathcal{H}(\zeta) \frac{\partial x}{\partial t}(\zeta, t)\,dz \\
= \frac{1}{2} \int_a^b \left( P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x)(\zeta, t) \right)^T \mathcal{H}(\zeta)x(\zeta, t)\,d\zeta + \\
\frac{1}{2} \int_a^b x(\zeta, t)^T \mathcal{H}(\zeta, t)(P_1 \frac{\partial}{\partial \zeta} (\mathcal{H}x)(\zeta, t)\,d\zeta \\
= \frac{1}{2} \int_a^b \frac{\partial}{\partial \zeta} \left[ (\mathcal{H}x)^T(\zeta, t)P_1(\mathcal{H}x)(\zeta, t) \right] \,d\zeta \\
= \frac{1}{2} \left[ (\mathcal{H}x)^T(\zeta, t)P_1(\mathcal{H}x)(\zeta, t) \right]_a^b,
\]

where we have used the symmetry of \(P_1\) and \(\mathcal{H}\).
So we have that the time-change of the energy/Hamiltonian satisfies

$$\frac{dE}{dt}(t) = \frac{1}{2} \left[ (\mathcal{H}x)^T (\zeta, t) P_1 (\mathcal{H}x) (\zeta, t) \right]^b_a.$$  \hspace{1cm} (6)

That is the change of internal energy goes via the boundary.

Note that we only used that $P_1, \mathcal{H}(\zeta)$ were symmetric. We did not need the specific form of $P_1$ or $\mathcal{H}$.

The balance equation (6) also holds for the system

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta, t) + P_0 [\mathcal{H}x] (\zeta, t)$$  \hspace{1cm} (7)

with $P_0$ anti-symmetric, i.e., $P_0^T = -P_0$.

Many systems can be written in this format.
The Stokes-Dirac structure associated with

\[
\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial \mathcal{H} x}{\partial \zeta}(\zeta, t) + P_0 \left[ \mathcal{H} x \right](\zeta, t)
\]

is given by

\[
\mathcal{D} = \left\{ \begin{pmatrix} f \\ f_\partial \end{pmatrix}, \begin{pmatrix} e \\ e_\partial \end{pmatrix} \in \mathcal{F} \times \mathcal{E} \mid f = P_1 \frac{\partial e}{\partial \zeta} + P_0 e, \\
\quad f_\partial = \frac{1}{\sqrt{2}} P_1 (e(b) - e(a)), e_\partial = \frac{1}{\sqrt{2}} (e(b) + e(a)) \right\}
\]
1.3 More examples

Example

Wave equation for the vibrating string:
\[
\frac{\partial^2 w}{\partial t^2} (\zeta, t) = \frac{1}{\rho(\zeta)} \frac{\partial}{\partial \zeta} \left[ T(\zeta) \frac{\partial w}{\partial \zeta} (\zeta, t) \right].
\]

where \( \rho(\zeta) \) is the mass density, and \( T(\zeta) \) is Young’s modulus.

As state variables, we introduce \( x_1 = \rho \frac{\partial w}{\partial t} \) (the momentum) and \( x_2 = \frac{\partial w}{\partial \zeta} \) (the strain).

Now the wave equation is in our format with
\[
P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}.
\]
Example

The model of Timoshenko beam is given

\[
\rho(\zeta) \frac{\partial^2 w}{\partial t^2}(\zeta, t) = \frac{\partial}{\partial \zeta} \left[ K(\zeta) \left( \frac{\partial w}{\partial \zeta}(\zeta, t) - \phi(\zeta, t) \right) \right]
\]

\[
I_{\rho}(\zeta) \frac{\partial^2 \phi}{\partial t^2}(\zeta, t) = \frac{\partial}{\partial \zeta} \left[ EI(\zeta) \frac{\partial \phi}{\partial \zeta} \right] + K(\zeta) \left( \frac{\partial w}{\partial \zeta}(\zeta, t) - \phi \right),
\]

where \( w(\zeta, t) \) is the transverse displacement of the beam and \( \phi(\zeta, t) \) is the rotation angle of a filament of the beam. The positive coefficients \( \rho(\zeta), I_{\rho}(\zeta), E(\zeta), I(\zeta), \) and \( K(\zeta) \) are the mass per unit length, the rotary moment of inertia of a cross section, Young’s modulus of elasticity, the moment of inertia of a cross section, and the shear modulus respectively.
By introducing the state variables

\[ x_1 = \frac{\partial w}{\partial \zeta} - \phi : \quad \text{shear displacement}, \]

\[ x_2 = \rho \frac{\partial w}{\partial t} : \quad \text{transverse momentum distribution}, \]

\[ x_3 = \frac{\partial \phi}{\partial \zeta} : \quad \text{angular displacement}, \]

\[ x_4 = I \rho \frac{\partial \phi}{\partial t} : \quad \text{angular momentum distribution}, \]

we can write this in our standard format, with

\[
P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\]
and

\[ \mathcal{H}(\zeta) = \text{diag}\{K(\zeta), \frac{1}{\rho(\zeta)}, EI(\zeta), \frac{1}{I_\rho(\zeta)}\}. \]
Example

Consider transmission lines in a network

\[
\begin{array}{c}
V \\
I
\end{array}
\quad K
\quad
\begin{array}{c}
V \\
I
\end{array}
\]

In the coupling parts $K$, we have that Kirchhoff laws holds. Hence charge flowing out of the transmission line I, enters II and III, etc.

The $P_1$ of the big system is the diagonal matrix, build from the uncoupled $P_1$’s (which are all the same). The $\mathcal{H}$ of the coupled system is the diagonal matrix of the uncoupled $\mathcal{H}$.

The coupling is written down as boundary conditions of the pde.
2 Outline of the course

For the first part of our course, the class of pde’s will be

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta, t) + P_0 [\mathcal{H}x](\zeta, t)$$

with the energy/Hamiltonian, i.e., “$H(x) = E(t)$”

$$E(t) = \frac{1}{2} \int_{a}^{b} x(\zeta, t)^T \mathcal{H}(\zeta) x(\zeta, t) d\zeta.$$ 

We assume that $P_1$ and $\mathcal{H}$ are real symmetric matrices, and $P_0$ is an anti-symmetric matrix, i.e., $P_0^T = -P_0.$
Since the power flow is via the boundary, it is logical that we want to control and observe the system via the boundary. In order to so, we need that the pde has a (unique) solution for any initial condition. This is only one of the questions we shall address in this course. Other questions are
• Uniqueness and existence of solutions for zero boundary conditions.

• Uniqueness and existence of solutions for (smooth) boundary control.

• Definition of outputs as boundary observations.

• Well posedness of the pde as a system.
3 Solutions for zero boundary conditions
For the solutions of pde’s the following holds

- Expression for the solutions are rare.
- Existence of solutions (for linear pde’s) can be shown more generally.
- What are solutions, classical/weak?

For existence of solutions to a pde, boundary conditions are essential. So we need to pose boundary conditions.
So the question is when does the linear pde

\[
\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial H x}{\partial \zeta}(\zeta, t) + P_0 [H x](\zeta, t)
\]

with boundary conditions

\[
\tilde{W}_B \begin{bmatrix}
[H x](b, t) \\
[H x](a, t)
\end{bmatrix} = 0
\]

have a (unique) solution?

To answer this question, we use “semigroup theory”.
3.1 Abstract differential equations

Consider the simple pde

\[
\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t).
\]

with boundary condition

\[ x(1, t) = 0. \]
We want that for every $t \geq 0 \, x(\cdot, t)$ lies in a state space. Thus this state space must be a function space, e.g. $L^2(0, 1)$.

So this implies that we “$t$” is more important than “$\zeta$”. So we write $x(t)$, and hide the spatial variable $\zeta$.

This implies that we have to read $\dot{x}(t)$ as

$$ (\dot{x}(t))(\zeta) = \frac{\partial x}{\partial t}(\zeta, t). $$

For a fixed (smooth) function $f \in X = L^2(0, 1)$ with $f(1) = 0$ we define

$$ (Af)(\zeta) = \frac{\partial f}{\partial \zeta}(\zeta). $$
So we have rewritten the pde

\[
\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t).
\]

with boundary condition

\[x(1, t) = 0\]

as

\[\dot{x}(t) = Ax(t), \quad x(0) = x_0\]

on the state space \(X = L^2(0, 1)\).
3.2 Semigroups

Throughout this course, we assume that

- $X$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.
- $\mathcal{L}(X)$ denotes the set of linear and bounded operators from $X$ to $X$. 
**Definition**

A strongly continuous semigroup is an operator-valued function from \([0, \infty)\) to \(L(X)\) which satisfies

- \(T(0) = I\)
- \(T(t)T(s) = T(t + s), \quad t, s \in [0, \infty)\)
- For all \(x_0 \in X\) there holds
  \[
  \lim_{t \downarrow 0} T(t)x_0 = x_0.
  \]

**Notation:** \((T(t))_{t \geq 0}\); **Short:** \(C_0\)-semigroup.
To motivate this definition, think of

\[ x_0 \mapsto T(t)x_0 \]

as a solution of a time-invariant, linear differential equation.

- \( T(0) = I \) \hspace{1cm} Trivial.
- For fixed \( t \), \( T(t) \) is linear \hspace{1cm} Is linearity of diff. eq.
- \( T(t)T(s) = T(t+s) \) \hspace{1cm} Is time invariance.
- \( T(t)x_0 \rightarrow x_0 \) if \( t \downarrow 0 \) \hspace{1cm} Is strong continuity.
Example

Let $A$ be an $n \times n$-matrix, then $e^{At}$ is a $C_0$-semigroup on $\mathbb{C}^n$ or $\mathbb{R}^n$.

For instance, if

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix},$$

then

$$e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}.$$
3.3 Generators

Consider the finite-dimensional semigroup

\[ e^{At} = \begin{pmatrix} e^t & e^{3t} - e^t \\ 0 & e^{3t} \end{pmatrix}. \]

**Question**

What is \( A \)?
Answer

Evaluate the derivative of the semigroup at $t = 0$.

Since $\frac{d}{dt}e^{At} = Ae^{At}$, we have

$$\frac{d}{dt}e^{At} \mid_{t=0} = A.$$

So given the (general) $C_0$-semigroup $(T(t))_{t \geq 0}$, we could try to find $A$ by differentiating it at $t = 0$. However, in general $(T(t))_{t \geq 0}$ is only (strongly) continuous.
Definition

If the following limit exists,

$$\lim_{t \downarrow 0} \frac{T(t)x_0 - x_0}{t},$$

then $x_0$ is in the domain of $A$, $D(A)$.

Furthermore, for $x_0 \in D(A)$, we define

$$Ax_0 = \lim_{t \downarrow 0} \frac{T(t)x_0 - x_0}{t}.$$

$A$ is named the infinitesimal generator of the $C_0$-semigroup $(T(t))_{t \geq 0}$. 

□
The following property holds for $A$ and $(T(t))_{t \geq 0}$.

**Lemma**
If $x_0 \in D(A)$, then $T(t)x_0 \in D(A)$, and

$$
\frac{d}{dt} [T(t)x_0] = AT(t)x_0.
$$

Hence for $x_0 \in D(A)$, we have that $x(t) := T(t)x_0$ is a (classical) solution of the abstract differential equation

$$
\dot{x}(t) = Ax(t) \quad x(0) = x_0.
$$
We summarize what we found until now

- A pde can be written as an abstract differential equation

\[ \dot{x}(t) = Ax(t) \]

on a state space \( X \).

- If there is a semigroup related to \( A \), i.e., \( A \) is its infinitesimal generator, then \( x(t) = T(t)x_0 \) is a solution.
3.4 Which $A$ generates a $C_0$-semigroup?

So we have that every semigroup has an infinitesimal generator, but you would like to know which operator $A$ generates a $C_0$-semigroup.

The answer is given by the Hille-Yosida Theorem.
3.5 Contraction semigroup

Definition

The $C_0$-semigroup $(T(t))_{t \geq 0}$ is contraction semigroup if

$$\|T(t)x_0\| \leq \|x_0\| \quad \text{for all } t \geq 0.$$

What can we say about these semigroups?
\[ \| T(t)x_0 \| \]

\[ \| x_0 \| \]

\[ t \rightarrow \]
We know that
\[ \| T(t)x_0 \|^2 = \langle T(t)x_0, T(t)x_0 \rangle. \]

For \( x_0 \in D(A) \), we have that the derivative of \( T(t)x_0 \) equals \( AT(t)x_0 \).

So if we differentiate \( \| T(t)x_0 \|^2 \), we find
\[
\frac{d}{dt} \| T(t)x_0 \|^2 = \langle AT(t)x_0, T(t)x_0 \rangle + \langle T(t)x_0, AT(t)x_0 \rangle.
\]
At time equal to zero, we find

\[ \frac{d}{dt} \left( \| T(t)x_0 \|^2 \right) \bigg|_{t=0} = \langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle. \]

We know that at \( t = 0 \), \( \| T(t)x_0 \| = \| x_0 \| \), and so if \( T(t) \) is a contraction semigroup, then

\[ \langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle = \frac{d}{dt} \| T(t)x_0 \|^2 \bigg|_{t=0} \leq 0. \]

This holds for all \( x_0 \in D(A) \).
Theorem (Lumer-Phillips)

Let $A$ be a densely defined operator, then $A$ generates a contraction semigroup on $X$ if and only if

1. $\langle Ax_0, x_0 \rangle + \langle x_0, Ax_0 \rangle \leq 0$ for all $x_0 \in D(A)$.

2. The range of $A - I$ is the whole of $X$.
Example

Consider $A$ which is given as

$$Ax = \frac{dx}{d\zeta}, \quad \zeta \in [0, 1]$$

with the domain

$$D(A) = \left\{ x \in L^2(0, 1) \mid x \text{ is absolutely continuous,} \right. \left. \frac{dx}{d\zeta} \in L^2(0, 1) \text{ and } x(1) = 0 \right\}.$$

Let us check the properties

- $A$ is densely defined in $L^2(0, 1)$. 
\[ \langle Ax, x \rangle + \langle x, Ax \rangle \]

\[ = \int_0^1 \frac{dx}{d\zeta}(\zeta)x(\zeta)d\zeta + \int_0^1 x(\zeta)\frac{dx}{d\zeta}(\zeta)d\zeta \]

\[ = \int_0^1 \frac{d}{d\zeta} [x(\zeta)x(\zeta)] d\zeta \]

\[ = |x(\zeta)|^2 \bigg|_0^1 \]

\[ = 0 - |x(0)|^2 \leq 0. \]
To see if the range of \((A - I)\) is everything, we have for every \(f \in L^2(0, 1)\) to solve \((A - I)x = f\) with \(x \in D(A)\).

This means solving

\[
\frac{dx}{d\zeta}(\zeta) - x(\zeta) = f(\zeta), \quad \zeta \in (0, 1)
\]

with boundary condition \(x(1) = 0\).

The solution of this differential equation with the given boundary value is

\[
x(\zeta) = -\int_\zeta^1 e^{\zeta-\tau} f(\tau) d\tau.
\]
Hence there is a semigroup associated to this $A$. It is not hard to see that it is the left shift semigroup.

$$(T(t)f)(\zeta) = \begin{cases} 
  f(t + \zeta) & \zeta + t \leq 1 \\
  0 & \zeta + t > 1 
\end{cases}$$
3.6 Homogeneous port-Hamiltonian systems

Now we return to the pde

\[
\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta, t) + P_0 [\mathcal{H}x](\zeta, t)
\]

with boundary conditions

\[
\tilde{W}_B \begin{bmatrix} [\mathcal{H}x](b, t) \\ [\mathcal{H}x](a, t) \end{bmatrix} = 0.
\]
Assumption

We assume the following:

• $P_1 = P_1^T$ and $P_1$ is invertible;

• $P_0 = -P_0^T$.

• $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^T$ and there exists $m, M > 0$ such that for all $\zeta \in [a, b]$, $mI \leq \mathcal{H}(\zeta) \leq MI$. 
As state space $X$ we choose our energy space, i.e.,

$$X = L^2([a, b]; \mathbb{R}^n)$$

with inner product (Why?)

$$\langle f, g \rangle_X = \frac{1}{2} \int_a^b g(\zeta)^T \mathcal{H}(\zeta) f(\zeta) d\zeta.$$
To use the abstract results, we define $A$ as

$$ Ax := P_1 \frac{d}{d\zeta} (\mathcal{H}x) + P_0 (\mathcal{H}x) $$

with domain

$$ D(A) = \{ x \in X \mid \mathcal{H}x \in H^1([a, b]; \mathbb{R}^n), \text{ and } \tilde{W}_B \begin{bmatrix} [\mathcal{H}x](b) \\ [\mathcal{H}x](a) \end{bmatrix} = 0 \}, $$

where $H^1$ is the Sobolev space

$$ H^1([a, b]; \mathbb{R}^n) = \{ f \in L^2([a, b]; \mathbb{R}^n) \mid f \text{ is absolutely continuous and } \frac{df}{d\zeta} \in L^2([a, b]; \mathbb{R}^n) \}. $$
Theorem

Consider the operator $A$ defined in (8) and (9) associated to a port-Hamiltonian system, that is, the assumptions on page 51 are satisfied. Furthermore, $\tilde{W}_B$, is an $n \times 2n$ matrix of rank $n$. Then the following statements are equivalent.

1. $A$ is the infinitesimal generator of a contraction semigroup on $X$.

2. $\langle Ax, x \rangle_X \leq 0$ for every $x \in D(A)$.

3. For all $x \in D(A)$ there holds

$$
(\mathcal{H}x)^T (b) P_1 (\mathcal{H}x) (b) - (\mathcal{H}x)^T (a) P_1 (\mathcal{H}x) (a) \leq 0,
$$

see page 15.
We make the following remarks:

- Conditions 2. and 3. are essentially the same.
- So to show that $A$ generates a contraction semigroup, you only have to check the easy condition.
- The condition; $A - I$ has range equal to $X$, is hidden in the rank condition on $\tilde{W}_B$. 
Example

Consider the transmission line on the spatial interval \([a, b]\)

\[
\begin{align*}
V(a) & \quad I(a) \quad a \\
I(b) & \quad V(b) \quad b
\end{align*}
\]

\[
\begin{align*}
\frac{\partial Q(\zeta, t)}{\partial t} &= -\frac{\partial}{\partial \zeta} \frac{\phi(\zeta, t)}{L(\zeta)} \\
\frac{\partial \phi(\zeta, t)}{\partial t} &= -\frac{\partial}{\partial \zeta} \frac{Q(\zeta, t)}{C(\zeta)}.
\end{align*}
\]

Here \(Q(\zeta, t)\) denotes the charge and \(\phi(\zeta, t)\) denotes the flux at position \(x\) and time \(t\). The voltage and current are given by \(V = Q/C\) and \(I = \phi/L\).
For this example we have that

\[(\mathcal{H}x)^T (b) P_1 (\mathcal{H}x) (b) - (\mathcal{H}x)^T (a) P_1 (\mathcal{H}x) (a) \]

\[= V(a) I(a) - V(b) I(b). \]

Hence if we have two (independent) boundary conditions such that

\[V(a) I(a) - V(b) I(b) \leq 0 \]

then this pde possesses a unique solution which is non-increasing in energy.
4 Exponential Stability
Now we know that there exist solutions which are non-increasing in energy, it is interesting to asked when do we have exponential stability.

We consider the same pde and assume the same conditions. Thus

$$\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial \mathcal{H}x}{\partial \zeta}(\zeta, t) + P_0 [\mathcal{H}x](\zeta, t)$$

with boundary conditions

$$\tilde{W}_B \begin{bmatrix} [\mathcal{H}x](b, t) \\ [\mathcal{H}x](a, t) \end{bmatrix} = 0.$$
Theorem

If \( \tilde{W}_B \) has rank \( n \), and there exists a \( \alpha > 0 \) such that one of the following two conditions is satisfied for all \( x \in D(A) \)

\[
(\mathcal{H}x)^T (b) P_1 (\mathcal{H}x) (b) - (\mathcal{H}x)^T (a) P_1 (\mathcal{H}x) (a) \\
\leq -\alpha \| (\mathcal{H}x) (a) \|^2, \tag{10}
\]

\[
(\mathcal{H}x)^T (b) P_1 (\mathcal{H}x) (b) - (\mathcal{H}x)^T (a) P_1 (\mathcal{H}x) (a) \\
\leq -\alpha \| (\mathcal{H}x) (b) \|^2, \tag{11}
\]

then the homogeneous solutions are exponentially stable.
Example

We consider the transmission line with the boundary conditions

\[ V(a) = 0 \text{ and } V(b) = RI(b) \text{ with } R > 0. \]

We have seen that

\[
\begin{align*}
(\mathcal{H}x)^T(b)P_1(\mathcal{H}x)(b) &- (\mathcal{H}x)^T(a)P_1(\mathcal{H}x)(a) \\
&= V(a)I(a) - V(b)I(b) \\
&= -RI(b)^2 \\
&= -\frac{R}{1 + R^2} \left\| \begin{pmatrix} V(b) \\ I(b) \end{pmatrix} \right\|^2,
\end{align*}
\]

where we used that

\[ V(b)^2 + I(b)^2 = (R^2 + 1)I(b)^2. \]

Hence putting a resistor at one end stabilizes the system exponentially.

\[ \square \]
5 Well-posedness
5.1 Introduction

The aim of this lecture is to show that general inputs and boundary conditions are possible for our class of port-Hamiltonian systems. We derive simple conditions for our results, which can be formulated as matrix conditions.
Consider the controlled transport equation on the interval \([0, 1]\) with scalar control and observation on the boundary

\[
\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial x}{\partial \zeta}(\zeta, t), \quad x(\zeta, 0) = x_0(\zeta), \quad \zeta \in [0, 1] \tag{12}
\]

\[
u(t) = x(1, t), \tag{13}
\]

\[
y(t) = x(0, t). \tag{14}
\]

The weak solution of (12)–(13) is given by

\[
x(\zeta, t) = \begin{cases} 
x_0(\zeta + t) & \zeta + t \leq 1 \\
u(\zeta + t - 1) & \zeta + t > 1. \end{cases} \tag{15}
\]
The state space is given by $X = L^2(0, 1)$.

We see that for every $x_0 \in X$ and every $u \in L^2(0, t_f)$ the state at time $t$

$$x(\zeta, t) = \begin{cases} x_0(\zeta + t) & \zeta + t \leq 1 \\ u(\zeta + t - 1) & \zeta + t > 1. \end{cases}$$

lies in $X$.

Furthermore, $x(t)$ is continuous, $\|x(\cdot, t + h) - x(\cdot, t)\| \to 0$ when $h \to 0$. 
From the output equation, \( y(t) = x(0, t) \), and the formula for the state trajectory, we see that

\[
y(t) = \begin{cases} 
  x_0(t) & t \leq 1 \\
  u(t - 1) & t > 1.
\end{cases}
\]

If \( x_0 \in X = L^2(0, 1) \) and \( u \in L^2(0, t_f) \), then \( y \in L^2(0, t_f) \).
Concluding, we see that for this example we have for every $x_0 \in X$ and every $u \in L^2$ the solution of our pde satisfies:

- For all $t > 0$, the state $x(t) \in X$, and $x(t)$ is continuous;
- $y \in L^2(0, t_f)$.

This is called well-posed.

In this lecture, we want to show that our systems with boundary input and output are well-posed.
5.2 Well-posed port-Hamiltonian systems

Consider our "standard" system with boundary control and observation

\[
\frac{\partial x}{\partial t}(\zeta, t) = P_1 \frac{\partial}{\partial \zeta} [\mathcal{H}x(t)] + P_0 [\mathcal{H}x(t)]
\]  

(16)

\[
u(t) = W_{B,1} \begin{pmatrix} [\mathcal{H}x](b, t) \\
[\mathcal{H}x](a, t) \end{pmatrix}
\]

(17)

\[
0 = W_{B,2} \begin{pmatrix} [\mathcal{H}x](b, t) \\
[\mathcal{H}x](a, t) \end{pmatrix}
\]  

(18)

\[
y(t) = W_C \begin{pmatrix} [\mathcal{H}x](b, t) \\
[\mathcal{H}x](a, t) \end{pmatrix}
\]  

(19)
The state space is $X = L^2((a, b); \mathbb{R}^n)$ with the (energy) norm

$$\| f \|_X^2 = \int_a^b f(\zeta)^T \mathcal{H}(\zeta) f(\zeta) d\zeta.$$ 

For this system and this norm, we define well-posedness.

Note that

$$\tilde{W}_B := \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}.$$
Definition

Consider the system (16)–(19) and let $k$ be the dimension of $u$. This system is well-posed if there exists a $t_f > 0$ and $m_f \geq 0$ such that the following holds:

1. The operator $A$ defined as $P_1 \frac{\partial}{\partial \zeta} \mathcal{H} + P_0 \mathcal{H}$ with domain,

$$D(A) = \{ x_0 \in X \mid \mathcal{H} x_0 \in H^1((a, b); \mathbb{R}^n), \quad \tilde{W}_B \left( \begin{bmatrix} \mathcal{H} x_0(b) \\ \mathcal{H} x_0(a) \end{bmatrix} \right) = 0 \}$$

is the infinitesimal generator of a $C_0$-semigroup on $X$. 
2. The following inequality holds for all $\mathcal{H}x_0 \in H^1((a, b); \mathbb{R}^n)$ and $u \in C^2([0, t_f); \mathbb{R}^k)$ with $u(0) = W_{B,1} \left( \begin{bmatrix} [\mathcal{H}x_0(b) \\ [\mathcal{H}x_0(a) \end{bmatrix} \right)$, and

$$
0 = W_{B,2} \left( \begin{bmatrix} [\mathcal{H}x_0(b) \\ [\mathcal{H}x_0(a) \end{bmatrix} \right)
$$

$$
\|x(t_f)\|_X^2 + \int_0^{t_f} \|y(t)\|^2 dt \leq m_f \left[ \|x_0\|_X^2 + \int_0^{t_f} \|u(t)\|^2 dt \right]
$$

(20)

We regard (20) as an energy relation.
We make the following remarks:

- The first item tells us that the homogeneous equation is well-defined. That is for every initial condition in our state space, there exists a unique solution with values in the state space.

- The second items tells us that we have a solution, if we start smoothly, but we can extend this solution to all square integrable inputs and all initial conditions.
Finally, we need the following definition.

**Definition**

Let $G(s)$ be the transfer function of (16)–(19). The system (16)–(19) is regular when $\lim_{s \in \mathbb{R}, s \to \infty} G(s)$ exists. If the system (16)–(19) is regular, then the feed-through term $D$ is defined as

$$D = \lim_{s \in \mathbb{R}, s \to \infty} G(s).$$
Theorem
Consider the partial differential equation (16)–(19) on the spatial interval \([a, b]\), with \(x(\zeta, t)\) taking values in \(\mathbb{R}^n\). Let \(X\) be the state space. Furthermore, assume that

- \(P_1\) is real-valued, invertible, and symmetric, i.e., \(P_1^T = P_1\),
- \(\mathcal{H}(\zeta)\) is a (real) symmetric matrix satisfying
  \[0 < mI \leq \mathcal{H}(\zeta) \leq M I, \zeta \in [a, b].\]
- \(P_1\mathcal{H}\) can be written as
  \[P_1\mathcal{H}(\zeta) = S^{-1}(\zeta)\Delta(\zeta)S(\zeta),\] (21)
  with \(\Delta(\cdot)\) diagonal, and both \(\Delta(\cdot)\) and \(S(\cdot)\) are continuously differentiable,
- \(\tilde{W}_B := \begin{pmatrix} W_{B,1} \\ W_{B,2} \end{pmatrix}\) is a \(n \times 2n\) matrix with rank \(n\),
• \( \text{rank} \begin{pmatrix} W_{B,1} \\ W_{B,2} \\ W_C \end{pmatrix} = n + \text{rank} (W_C). \)

If the homogeneous pde, i.e., \( u \equiv 0 \), generates a \( C_0 \)-semigroup on \( X \), then the system (16)–(19) is well-posed, and the corresponding transfer function \( G \) is regular. Furthermore, we have that

\[
\lim_{\text{Re}(s) \to \infty} G(s) = \lim_{s \to \infty, s \in \mathbb{R}} G(s).
\]
Remark

We can make the following remarks concerning the conditions in our theorem.

• The first two conditions are very standard, and are assumed to be satisfied for all our port-Hamiltonian systems until now.

• Note that we do not have a condition on $P_0$. In fact the term $P_0 \mathcal{H}$ may be replaced by any bounded operator on $X$.

• The third condition is not very strong, and will almost always be satisfied if $\mathcal{H}(\cdot)$ is continuously differentiable. Note that $\Delta$ contains the eigenvalues of $P_1 \mathcal{H}$, whereas $S^{-1}$ contains the eigenvectors.

• The fourth condition tells us that we have $n$ boundary conditions, when we put the input to zero. This very logical, since we have an $n$’th order pde.
• The last condition states that we are not measuring quantities that are set to zero, or set to be an input. This condition is not important for the proof, and will normally follow from correct modeling.
The proof of this theorem is not very difficult, but needs many steps. In order that we see what we are doing we present the proof for an example. We choose the example of the vibrating string.
5.3 Proving the theorem for the vibrating string.

Consider the vibrating string

$$\frac{\partial}{\partial t} \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \zeta} \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix} \begin{pmatrix} x_1(\zeta, t) \\ x_2(\zeta, t) \end{pmatrix}, \quad (22)$$

where $x_1 = \rho \frac{\partial w}{\partial t}$ is the momentum and $x_2 = \frac{\partial w}{\partial \zeta}$ is the strain.
As control and observation we choose

\[ u_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(b, t) \\ x_2(a, t) \end{pmatrix} \]  \hspace{1cm} (23)

and as output

\[ y_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(a, t) \\ x_2(b, t) \end{pmatrix}. \]  \hspace{1cm} (24)
First we check whether the conditions of the theorem are satisfied.

We have that

\[
P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{H}(\zeta) = \begin{pmatrix} \frac{1}{\rho(\zeta)} & 0 \\ 0 & T(\zeta) \end{pmatrix}.
\]

These are symmetric, and \( \mathcal{H} \) is positive.
Furthermore, we have that

\[ P_1 \mathcal{H}(\zeta) = \begin{pmatrix} 0 & T(\zeta) \\ \frac{1}{\rho(\zeta)} & 0 \end{pmatrix} \]

satisfies the third condition, since

\[ P_1 \mathcal{H} = S^{-1} \Delta S = \begin{pmatrix} \gamma & -\gamma \\ \frac{1}{\rho} & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \begin{pmatrix} \frac{1}{2\gamma} & \frac{\rho}{2} \\ -\frac{1}{2\gamma} & \frac{\rho}{2} \end{pmatrix}, \]

with \( \gamma(\zeta) = \frac{T(\zeta)}{\rho(\zeta)}. \)
We have that \( n \) is 2, and it is easy to see that the other conditions are satisfied.

Note that \( W_{B,2} = 0 \).
We want to show that the above system is well-posed.

The first step which we make is that we change our state,

\[ \tilde{x} = Sx. \]

First we see what this change of variables does with our pde. We find

\[
\frac{\partial \tilde{x}}{\partial t} = \frac{\partial Sx}{\partial t} = S \frac{\partial x}{\partial t} = S \frac{\partial P_1 \mathcal{H} x}{\partial \zeta}
\]

\[
= S \frac{\partial S^{-1} \Delta Sx}{\partial \zeta}
\]

\[
= \frac{\partial \Delta \tilde{x}}{\partial \zeta} - \dot{S} S^{-1} \Delta \tilde{x}.
\]
If we ignore the term, $\dot{S}S^{-1} \Delta \tilde{x}$, then we have a very simple pde;

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{x}_1(\zeta, t) \\ \tilde{x}_2(\zeta, t) \end{pmatrix} = \frac{\partial \tilde{x}}{\partial t} = \frac{\partial \Delta \tilde{x}}{\partial \zeta} = \frac{\partial}{\partial \zeta} \begin{pmatrix} \gamma(\zeta)\tilde{x}_1(\zeta, t) \\ -\gamma(\zeta)\tilde{x}_2(\zeta, t) \end{pmatrix}.$$  

Hence, we have two transport equations, one with positive, and the other with negative velocity. These pde's can be solved.
Lemma

Let $\lambda(\zeta)$ be a positive continuous differentiable function on the interval $[a, b]$. With this function we define the scalar system on $\zeta \in [a, b]$.

$$\frac{\partial w}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta} (\lambda(\zeta)w(\zeta, t)), \quad w(\zeta, 0) = w_0(\zeta).$$

(25)

The value at $b$ we choose as input

$$u(t) = \lambda(b)w(b, t)$$

(26)

and as output we choose the value on the other end

$$y(t) = \lambda(a)w(a, t).$$

(27)
The system (25)–(27) is a well-posed system on the state space $L^2(a, b)$. Its transfer function is given by

$$G(s) = e^{-p(b)s}, \quad (28)$$

where $p$ is defined as

$$p(\zeta) = \int_a^\zeta \lambda(\zeta)^{-1} d\zeta \quad \zeta \in [a, b]. \quad (29)$$

This transfer function satisfies

$$\lim_{\text{Re}(s) \to \infty} G(s) = 0. \quad (30)$$
The proof is based on the balance equation. The system
\[
\frac{\partial w}{\partial t} = \frac{\partial}{\partial \zeta} (\lambda w), \quad u(t) = \lambda(b)w(b, t), \quad y(t) = \lambda(a)w(a, t).
\]
is a standard port-Hamiltonian system, and so
\[
\frac{d}{dt} \int_a^b w(\zeta, t) \lambda(\zeta) w(\zeta, t) d\zeta = [[\lambda(\zeta)w(\zeta, t)] \lambda(\zeta)w(\zeta, t)]_a^b
\]
\[
= |u(t)|^2 - |y(t)|^2,
\]
Thus for all $t_f > 0$ we have that
\[
\int_a^b w(\zeta, t_f) \lambda(\zeta) w(\zeta, t_f) d\zeta - \int_a^b w(\zeta, 0) \lambda(\zeta) w(\zeta, 0) d\zeta
\]
\[
= \int_0^{t_f} |u(\tau)|^2 d\tau - \int_0^{t_f} |y(\tau)|^2 d\tau.
\]
If the $\lambda$ is negative, then $a$ and $b$ change places.

This implies that our system

$$
\frac{\partial}{\partial t} \begin{pmatrix} \tilde{x}_1(\zeta, t) \\ \tilde{x}_2(\zeta, t) \end{pmatrix} = \frac{\partial}{\partial \zeta} \begin{pmatrix} \gamma(\zeta)\tilde{x}_1(\zeta, t) \\ -\gamma(\zeta)\tilde{x}_2(\zeta, t) \end{pmatrix}.
$$

with input and output

$$
u_s(t) = \begin{pmatrix} \gamma(b)\tilde{x}_1(b, t) \\ -\gamma(a)\tilde{x}_2(a, t) \end{pmatrix} \quad y_s(t) = \begin{pmatrix} \gamma(a)\tilde{x}_1(a, t) \\ -\gamma(b)\tilde{x}_2(b, t) \end{pmatrix}
$$

is well-posed.
Now we return to our original system.

Our input and output were not given as $u_s$ and $y_s$, but as

$$u_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(b, t) \\ x_2(a, t) \end{pmatrix}, \quad y_1(t) = \begin{pmatrix} \frac{x_1}{\rho}(a, t) \\ x_2(b, t) \end{pmatrix}. $$

We can write this as linear combinations of $u_s$ and $y_s$. Namely,
\[ u_1(t) = \begin{pmatrix} \frac{1}{\rho(b)} & 0 \\ 0 & -\frac{1}{\sqrt{T(a)\rho(a)}} \end{pmatrix} u_s(t) + \begin{pmatrix} 0 & \frac{1}{\rho(b)} \\ \frac{1}{\sqrt{T(a)\rho(a)}} & 0 \end{pmatrix} y_s(t) \]

\[ y_1(t) = \begin{pmatrix} 0 & -\frac{1}{\rho(a)} \\ \frac{1}{\sqrt{T(b)\rho(b)}} & 0 \end{pmatrix} u_s(t) + \begin{pmatrix} \frac{1}{\rho(a)} & 0 \\ 0 & \frac{1}{\sqrt{T(b)\rho(b)}} \end{pmatrix} y_s(t) \]

\[ = K u_s(t) + Q y_s(t) \]

\[ = O_1 u_s(t) + O_2 y_s(t). \]

This we can write as a feedback loop.
Analysis of port-Hamiltonian Systems

Well-posedness

\[ u_1(t) \]

\[ K^{-1} \]

\[ u_s(t) \]

\[ G_s(s) \]

\[ y_s(t) \]

\[ O_1 \]

\[ O_2 \]

\[ y_1(t) \]
Now we have a theorem by George Weiss, telling that if
\[ \lim_{\text{Re}(s) \to \infty} G_s(s) = 0, \]
then the feedback gives a well-posed system.

Hence the only condition we have to check if \( K \) is invertible.
We have the following remarks:

- If $K$ is not invertible, then one can prove that you don’t have a $C_0$-semigroup.
- We have ignored the term $\dot{S} S^{-1} \Delta \tilde{x}$. This term does influence the expression of the solution, but not the existence, nor the well-posedness, nor the feed-through.
5.4 Summary

In this part we have seen the following

- Well-posedness is a natural property.

- For our class of port-Hamiltonian systems well-posedness is checkable by only checking the semigroup condition.

- It can even checked via a matrix condition.
6 Dissipation
Our pde’s with dissipation are assumed to be of the following form

\[ \frac{\partial x}{\partial t}(\zeta, t) = (\mathcal{J} - \mathcal{G}_R S \mathcal{G}_R^*) (\mathcal{H}x)(\zeta, t), \quad \zeta \in [a, b], \quad (31) \]

where

\[ \mathcal{J}x = P_1 \frac{\partial x}{\partial \zeta} + P_0 x, \quad \mathcal{G}_R f = G_1 \frac{\partial f}{\partial \zeta} + G_0 f, \quad (32) \]

\[ \mathcal{G}_R^* x = -G_1^T \frac{\partial x}{\partial \zeta} + G_0^T x, \quad S > 0. \quad (33) \]

Note that if \( S = 0 \) this becomes our original port-Hamiltonian system.
Introduce now the mapping $\mathcal{J}_e$ as

$$
\begin{pmatrix}
  f \\
  f_p
\end{pmatrix}
= \mathcal{J}_e
\begin{pmatrix}
  e \\
  e_p
\end{pmatrix}
= \begin{pmatrix}
  \mathcal{J} & G_R \\
  -G^*_R & 0
\end{pmatrix}
\begin{pmatrix}
  e \\
  e_p
\end{pmatrix}
$$

with the closure relation

$$
e_p = S f_p.
$$

Then

$$
f = \mathcal{J} e + G_R e_p = \mathcal{J} e + G_R S f_p = \left( \mathcal{J} - G_R S G^*_R \right) e
$$

Choosing $f = \frac{\partial x}{\partial t}$ and $e = \mathcal{H} x$ and we have our system.
Figure 1: Interconnection structure.
Note that $\mathcal{J}_e$ is given

$$
\begin{pmatrix}
\mathcal{J} & G_R \\
-G_R^* & 0
\end{pmatrix}
= \begin{pmatrix}
P_1 & G_1 \\
G_1^T & 0
\end{pmatrix}
\frac{\partial}{\partial \zeta}
+ \begin{pmatrix}
P_0 & G_0 \\
-G_0^T & 0
\end{pmatrix}

= P_{1,\text{ext}} \frac{\partial}{\partial \zeta} + P_{0,\text{ext}}
$$

Thus it is of our standard form.

Choosing $f = \frac{\partial x}{\partial t}$, $f_p = \frac{\partial x_p}{\partial t}$ and $e = \mathcal{H}x$, $e_p = x_p$ we know for which boundary conditions $\mathcal{J}_e$ generates a contraction semigroup.
**Theorem**

Consider the operator $\mathcal{J}_e \begin{pmatrix} \mathcal{H} & 0 \\ 0 & I \end{pmatrix}$. If this operator together with the boundary conditions

$$
\tilde{W}_B \begin{pmatrix} \mathcal{H}_x(b) \\ x_p(b) \\ \mathcal{H}_x(a) \\ x_p(a) \end{pmatrix} = 0
$$

generates a contraction semigroup on $X \oplus X_p$, then the operator

$$(\mathcal{J} - G_R S G_R^*) \mathcal{H}$$

with the boundary conditions

$$
\tilde{W}_B \begin{pmatrix} \mathcal{H}_x(b) \\ -(S G_R^*) \mathcal{H}_x(b) \\ \mathcal{H}_x(a) \\ -(S G_R^*) \mathcal{H}_x(a) \end{pmatrix} = 0
$$

generates a contraction semigroup on $X$. 

□
Example

We want to show that the diffusion equation

\[
\frac{\partial x}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta} \left[ \lambda(\zeta) \frac{\partial x}{\partial \zeta}(\zeta, t) \right], \quad \zeta \in [0, 1],
\]

with the boundary condition

\[
x(0) = x(1) = 0
\]

generates a contraction semigroup.
We know that $S = \lambda$, and the corresponding $J_e$ is given by

$$
\begin{pmatrix}
0 & \frac{\partial}{\partial \zeta} \\
\frac{\partial}{\partial \zeta} & 0
\end{pmatrix}
$$

For this operator, we know when it will generate a contraction semigroup on $L^2((0, 1); \mathbb{R}^2)$. It generates a contraction group when $x(0) = x(1) = 0$.

By our general theorem, we conclude that the heat equation with zero temperature at the boundary possesses a mild solution.
Remark

Please note that there is no relation between the solutions. We only proved that existence was related.
References


