Uniqueness of Tensor Decomposition

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Search for Latent Variables: ICA, Tensors, and NMF

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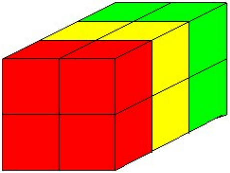
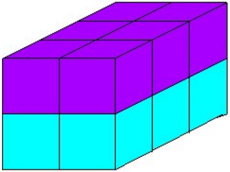
Tensors as multidimensional matrices

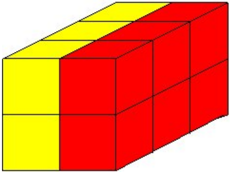
A (complex) $a \times b \times c$ tensor is an element of the space $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, it can be represented naively as a 3-dimensional matrix.

Here is a tensor of format $2 \times 2 \times 3$. 

Slices of tensors

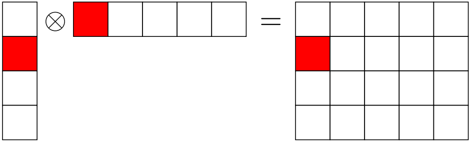
Several slices of a $2 \times 2 \times 3$ tensor.





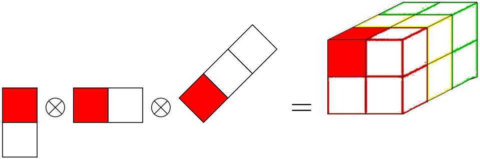
The decomposable (rank one) tensors

Here is a decomposable matrix



$a_{ij} = x_i y_j$

Here is a decomposable tensor



$a_{ijk} = x_i y_j z_k$

A (CP) decomposition of a tensor $T \in \mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ is

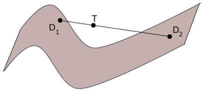
$$T = \sum_{i=1}^{r} D_i \qquad \text{(CANDECOMP, PARAFAC)}$$

with decomposable D_i and minimal r (called the rank). The variety of decomposable tensors is the Segre variety

$$X = \mathbb{P}(\mathbb{C}^a) \times \mathbb{P}(\mathbb{C}^b) \times \mathbb{P}(\mathbb{C}^c).$$

Geometric interpretation, secant varieties

$X = \mathbb{P}(\mathbb{C}^a) \times \mathbb{P}(\mathbb{C}^b) \times \mathbb{P}(\mathbb{C}^c)$.
The closure of the variety of tensors of rank $\leq k$ is called the k-secant variety of X and it is denoted by $\sigma_k(X)$.



Picture of a 2-secant.

We have the filtration

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \dots$$

Rank is difficult to be computed

In principle, to compute the (border) rank of a tensor T one has first to check the minimum k such that $T \in \sigma_k(X)$ in the filtration

$$X = \sigma_1(X) \subset \sigma_2(X) \subset \dots$$

Having equations of $\sigma_k(X)$, one can check if T satisfies these equations.

Unfortunately, the equations of $\sigma_k(X)$, despite being algebraic, look very difficult to be computed explicitly.

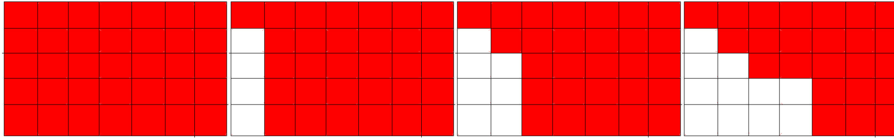
Matrix case, Gaussian elimination

The equations of the varieties of matrices of rank $\leq k$ are known, they are given by the $(k+1)$-minors of the matrix.

In practice, to detect the rank of a matrix, one uses directly Gaussian elimination, avoiding the explicit expressions of the minors.

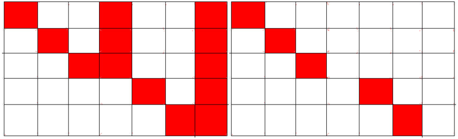
Gaussian elimination

Gaussian elimination consists in simplifying a matrix, by adding to a row a multiple of another one, and so on. This transformation corresponds to left multiplication by invertible matrices.



Gaussian elimination and canonical form

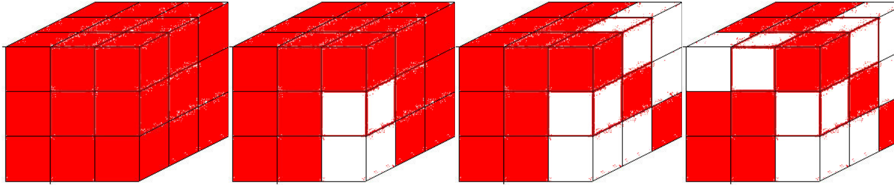
adding rows backwards....



...adding columns we get a canonical form ! This matrix of rank 5 is the sum of five rank one (or "decomposable") matrices.

Trying Gaussian elimination on a 3-dimensional tensor

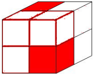
We can add a scalar multiple of a slice to another slice.

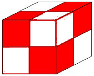


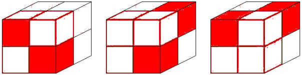
How many zeroes we may assume, at most ?

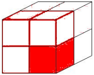
Strassen showed in 1983 one remains with at least 5 nonzero entries. Even, at least $5 (> 3)$ decomposable summands.

The six canonical forms of a $2 \times 2 \times 2$ tensor

 general rank 2 .

 hyperdeterminant vanishes.

 support on one slice (only not symmetric)!

 rank 1.

$2 \times 2 \times 2$ is one of the few lucky cases, where such a classification is possible.

Gaussian elimination and group action

A dimensional count shows that we cannot expect finitely many canonical forms.

- The dimension of $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$ is n^3.
- The dimension of $GL(n) \times GL(n) \times GL(n)$ is $3n^2$, *too small for* $n \geq 3$.

The same argument works for general $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$, $d \geq 3$, with a few small dimensional exceptions.

A disadvantage may be turned into an advantage for modeling.

The lack of canonical forms makes tensors with $d \geq 3$ modes interesting from other point of views. In some sense tensors with $d \geq 3$ encode more subtle properties that cannot be detected by linear change of coordinates. Geometers say that tensors with $d \geq 3$ modes have *moduli*. They are more *flexible* objects for modeling.

Basic question in this talk.

CP decomposition of matrices (tensors with $d = 2$ modes) is never unique.
What happens for $d \geq 3$ modes ?

Relevance of uniqueness in CP decomposition

A tensor T has a *unique* CP decomposition (of rank r) if all the decompositions $T = \sum_{i=1}^{r} a_i b_i c_i$ differ just be re-ordering the summands.
A tensor which has a unique CP decomposition is called *identifiable*.

Uniqueness of CP decomposition is a crucial property, needed in many applications, which allows to recover the individual summands from a tensor.

$$T = \sum_{i=1}^{r} a_i b_i c_i \qquad \Longrightarrow \qquad \{a_i b_i c_i\} \qquad ?$$

The Kruskal criterion

The well known Kruskal Criterion gives a sufficient condition which provides the identifiability of a given CP decomposition.

Theorem (Kruskal, 1977)

Let $T = \sum_{i=1}^{r} a_i b_i c_i$. Let k_A be the maximum m such that all subsets of m vectors taken from the list $\{a_1, \ldots, a_r\}$ are independent. Same for k_B, k_C.
If $r \leq \frac{1}{2}(k_A + k_B + k_C) - 1$ then $\mathrm{rk}(T) = r$ and the CP decomposition of T is unique.

Generic identifiability from Kruskal criterion

Definition

Generic k-identifiability holds for $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$ if the general tensor of rank k is identifiable.

Kruskal criterion answers affirmatively to generic k-identifiability question when the rank is relatively small.

Kruskal bound

Kruskal criterion provides generic k-identifiability for $n \times n \times n$ tensors when

$$k \leq \frac{3n}{2} - 1.$$

Kruskal criterion has a large amount of applications, but it is still unsatisfactory because Kruskal bound is too restrictive. The general $n \times n \times n$ tensors have a rank $\sim \frac{n^2}{3}$.

Derksen examples.

H. Derksen gives in 2013 some examples of CP decompositions with $\frac{1}{2}(k_A + k_B + k_C) - \frac{1}{2}$ summands which are not unique. So, regarding Kruskal criterion, the inequality provided by Kruskal (Kruskal bound) cannot be improved.

Despite this argument, we remark that Derksen's examples are not generic, and it is possible to improve further Kruskal bound for generic tensor of a given rank.

How tools from Algebraic Geometry can help for identifiability questions

Algebraic Geometry provides a necessary condition for generic k-identifiability, looking at the dimension of the secant variety.

If $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$ is generically k-identifiable then the dimension $\dim \sigma_k(X)$ of the k-th secant variety to the Segre variety X is equal to $\min\left(k(1+\sum_i (n_i-1))-1, (\prod_i n_i)-1\right)$.

Note that holds the inequality

$$\dim \sigma_k(X) \leq \min\left(k(1+\sum_i (n_i-1))-1, (\prod_i n_i)-1\right).$$

If $<$ holds (*defective cases*), generic identifiability cannot holds. Computation of dimension of secant variety becomes crucial.

Terracini Lemma

Terracini Lemma describes the tangent space at a secant variety.

Lemma (Terracini)

Let $z \in < x_1, \ldots, x_k >$ *be general. Then*

$$T_z\sigma_k(X) = < T_{x_1}X, \ldots, T_{x_k}X > .$$

It can be used to compute the dimension of secant variety at general (random) points.

Toward an Alexander-Hirschowitz Theorem in the non symmetric case

Known defective examples

Let $\dim V_i = n_i$, $n_1 \leq \ldots \leq n_d$, $X = \mathbb{P}^{n_1-1} \times \ldots \times \mathbb{P}^{n_d-1}$
Only known examples when
$\dim \sigma_k(X) < \min\left(k(1+\sum_i(n_i-1)) - 1, (\prod_i n_i) - 1\right)$ are

- unbalanced case, where $n_d \geq 2 + \prod_{i=1}^{d-1} n_i - \sum_{i=1}^{d-1}(n_i - 1)$,
- $k = 3$, $(n_1, n_2, n_3) = (3, m, m)$ with m odd [Strassen],
- $k = 3$, $(n_1, n_2, n_3) = (3, 4, 4)$, [Abo-O-Peterson],
- $k = 4$, $(n_1, n_2, n_3, n_4) = (2, 2, n, n)$.

Results in the general case

Theorem (Strassen-Lickteig)

There are no exceptions (no defective cases) for $\mathbb{P}^n \times \mathbb{P}^n \times \mathbb{P}^n$, *beyond the variety* $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$.

Theorem

- *The unbalanced case is completely understood [Catalisano-Geramita-Gimigliano].*
- *The known defective examples are the only ones in the cases:*
  (i) $\forall k$, $n_i = 1$, *binary case, [Catalisano-Geramita-Gimigliano]*
  (ii) $s \leq 55$ *[Vannieuwenhoven - Vanderbril - Meerbergen] (computation with large Terracini matrices)*

The contact locus

Chiantini and Ciliberto discovered in 2001 that a classical paper by Terracini from 1911 contained a clever idea which allows to treat identifiability by infinitesimal computations.

Any tensor $A \in \sigma_k(X)$ has a contact locus defined by

$$C_k(A) := \{x \in X | T_x X \subset T_A \sigma_k(X)\}.$$

Theorem (Chiantini-Ciliberto, Chiantini-O-Vannieuwenhoven)

If $A = \sum_{i=1}^{k} x_i$ has another different CP decomposition of rank k, AND if A is a smooth point in $\sigma_k(X)$, then $C_k(A)$ is positive dimensional at any x_i.

Note that the smoothness assumption is always satisfied for general points (tensors). On the contrary, it is a critical assumption for specific points (tensors).

A Linear Algebra algorithm for identifiability

The Chiantini-Ciliberto Criterion may be implemented in an algorithm which detects generic identifiability just by linear algebra, by computing the tangent space of the contact locus at a point x_1 appearing in $A = \sum_{i=1}^{k} x_i$. In practice we reduce to computing the rank of certain (large) Jacobian and Hessian-like matrices evaluated at x_1.

Criterion

If the tangent space of the contact locus at x_1 is zero dimensional, then we get k-identifiability.

The proof may be understood with the help of a picture.

The unreasonable effectiveness of contact locus

Infinitesimal computations in identifiability settings are counter-intuitive, because two different CP decompositions of the same tensor can be very far, one from each other.

Terracini method ("weak defectivity") allows to detect if a second CP decomposition may exists, just by infinitesimal computations "on a neighborhood of the first one".

Details of the algorithm

1. Pick r general decomposable tensors $a_ib_ic_i \in A \otimes B \otimes C$.
2. Compute cartesian equations $H_1, \ldots H_e$ for the linear subspace spanned by $Ab_ic_i + a_iBc_i + a_ib_iC$.
3. Compute all partial derivatives $\frac{\partial}{\partial a^i}, \frac{\partial}{\partial b^j}, \frac{\partial}{\partial c^k}$ of $H_s(abc)$.
4. Evaluate at $a_1b_1c_1$ the Jacobian matrix of the equations got in step 3.
5. If the rank of the Jacobian in step 4 is $\dim A + \dim B + \dim C - 3$ then tensors in $A \otimes B \otimes C$ are r-generically identifiable.

Running the algorithm for identifiability

Running this algorithm, in several collaborations with Bocci, Chiantini and Vannieuwenhoven, it has been discovered that generic k-identifiability still holds for all subgeneric ranks k, unless a list of exceptions (the weakly defective cases).

Uniqueness in the subgeneric case

The known examples

Assume for simplicity $d = 3$. Only known examples where the general $f \in V_1 \otimes V_2 \otimes V_3$ ($\dim V_i = n_i$) of subgeneric rank s has a NOT UNIQUE CP decomposition, besides the defective ones, are

- unbalanced case, rank $s = (n_1 - 1)(n_2 - 1) + 1$, $n_3 \geq (n_1 - 1)(n_2 - 1) + 2$
- rank 6 $(n_1, n_2, n_3) = (4, 4, 4)$ where there are exactly two CP decompositions. Here the contact locus is an elliptic curve.
- rank 8 $(n_1, n_2, n_3) = (3, 6, 6)$, where there are exactly six CP decompositions.

Running the algorithm for large cases

Theorem (Chiantini-O-Vannieuwenhoven)

The list of previous slide is complete if $n_1 n_2 n_3 \leq 15,000$.

Obtained by computing rank of large Hessian-like matrices. Couple of months computation, testing 75993 varieties.

Theoretical results on generic uniqueness

Theorem

There is a unique decomposition for general tensor of rank k in $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$, *with* $a \le b \le c \le k$,

- *if* $k \le \frac{a+b+c-2}{2}$ *[Kruskal, 1977]*
- *if* $k \le \frac{abc}{a+b+c-2} - c$, $3 \le a$ *[Bocci-Chiantini-O., Strassen]*
- *if* $2 \le a \le b \le c \le k$ *and* $k \le \frac{1}{2}\left(a + b + 2c - 2 - \sqrt{(a-b)^2 - 4c}\right)$ *[Domanov - De Lauthawer, 2014] .*

The contact locus for specific tensors

The main difficulty to prove uniqueness for specific tensors with the geometric approach is the checking of smoothness of the secant variety. It is paradoxical, because the general point of a given rank is smooth. This is related to our ignorance about equations of secant varieties.

Nevertheless, in many cases, equations of secant varieties are known and this algorithm can detect identifiability of specific tensors beyond Kruskal bound, but still in a range linear with n.

Comparison of identifiability criteria

Identifiability for specific tensors of rank r may be checked for $\mathbb{C}^a \otimes \mathbb{C}^b \otimes \mathbb{C}^c$ in the following cubic cases

(a, b, c)	Contact locus method	Kruskal	Domanov–De Lathauwer
$(4, 4, 4)$	$r \leq 4$	$r \leq \mathbf{5}$	$r \leq \mathbf{5}$
$(5, 5, 5)$	$r \leq \mathbf{7}$	$r \leq 6$	$r \leq 6$
$(6, 6, 6)$	$r \leq \mathbf{8}$	$r \leq \mathbf{8}$	$r \leq \mathbf{8}$
$(7, 7, 7)$	$r \leq \mathbf{11}$	$r \leq 9$	$r \leq 9$
$(8, 8, 8)$	$r \leq \mathbf{12}$	$r \leq 11$	$r \leq 11$
$(9, 9, 9)$	$r \leq \mathbf{15}$	$r \leq 12$	$r \leq 13$

In some unbalanced cases Domanov-De Lathauwer criterion is the best one.

Identifiability of a specific tensor

We consider the following rank-7 tensor $A \in \mathbb{C}^5 \otimes \mathbb{C}^5 \otimes \mathbb{C}^5$:

$$A = \begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} \otimes \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix} \otimes \begin{bmatrix}1\\5\\7\\-5\\-7\end{bmatrix} + \begin{bmatrix}4\\3\\2\\-1\\-2\end{bmatrix} \otimes \begin{bmatrix}11\\13\\12\\15\\14\end{bmatrix} \otimes \begin{bmatrix}-2\\6\\5\\-3\\6\end{bmatrix} + \sum_{i=1}^{5} e_i \otimes e_i \otimes e_i \tag{1}$$

with e_i the ith standard basis vector in $\mathbb{C}^5$.

This example is beyond Kruskal bound. Still it can be proved it is a smooth point of 7-th secant variety. The contact locus is zero dimensional and it reduces to the seven summands. Hence A is identifiable.

An example when the contact locus fails

A. Stegeman and J. M. F. Ten Berge (2006),
I. Domanov and L. De Lathauwer (2013) prove the identifiability of the rank 5 tensor in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^5$ given by the columns of the following matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 3 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix}, \; C = I_5.$$

The contact locus is one dimensional at just the fifth summand. So our geometric criterion does not apply here. This example is again beyond Kruskal bound.

Symmetric tensors = homogeneous polynomials

In the case $n_1 = \ldots = n_d = n$ we may consider symmetric tensors $f \in S^d\mathbb{C}^n$. They can be regarded as homogeneous polynomials of degree d in $x_1, \ldots x_n$.

$$f = \sum_{i=1}^{r} c_i (l_i)^d \qquad \text{with } l_i \in \mathbb{C}^n$$

with minimal r (symmetric rank).

Example: $7x^3 - 30x^2y + 42xy^2 - 19y^3 = (-x+2y)^3 + (2x-3y)^3$
$\mathrm{rk}\left(7x^3 - 30x^2y + 42xy^2 - 19y^3\right) = 2$

The variety of decomposable (rank one) tensors is the Veronese variety $v_d(\mathbb{P}^{n-1})$.

The Comon Conjecture

Comon Conjecture

Let t be a symmetric tensor. Are the rank and the symmetric rank of t equal ? Comon conjecture gives affirmative answer.

Known to be true for $t \in S^d\mathbb{C}^{n+1}$, when $n = 1$ or $d = 2$ and in few other cases.

Theorem (Campbell, [1891] Terracini, [1916] Alexander-Hirschowitz [1995])

Let $d \geq 3$. The k-th secant variety to the Veronese variety $v_d(\mathbb{P}^n)$ has dimension $\min\left(k(n+1)-1, \binom{n+d}{d}-1\right)$, *with the only exceptions (defective)*

- $\sigma_{n(n+3)/2} v_4(\mathbb{P}^n)$, $2 \leq n \leq 4$,
- $\sigma_7 v_3(\mathbb{P}^4)$.

The generic symmetric rank

The following corollary to AH Theorem is a friendly version in terms of rank.

Corollary (to AH Theorem)

Let $d \geq 3$. The general $f \in S^d\mathbb{C}^{n+1}$ ($d \geq 3$) has rank

$$\left\lceil \frac{\binom{n+d}{d}}{n+1} \right\rceil$$

which is called the generic rank, with the only exceptions

- $S^4\mathbb{C}^{n+1}$, $\quad 2 \leq n \leq 4$, *where the generic rank is* $\binom{n+2}{2}$.
- $S^3\mathbb{C}^5$, *where the generic rank is* 8.

The symmetric case: uniqueness in the subgeneric case

Theorem (Sylvester[1851], Chiantini-Ciliberto, Mella, Ballico, [2002-2005], Chiantini-O-Vannieuwenhoven [2015])

Let $d \geq 3$. The general $f \in S^d\mathbb{C}^{n+1}$ of rank s smaller than the generic one has a unique symmetric decomposition, with the only exceptions (called weakly defective)

- *the four Alexander-Hirschowitz exceptions, when there are infinitely many symmetric decompositions.*
- *rank 9 in $S^6\mathbb{C}^3$, where there are exactly two symmetric decompositions.*
- *rank 8 in $S^4\mathbb{C}^4$, where there are exactly two symmetric decompositions.*
- *rank 9 in $S^3\mathbb{C}^6$, where there are exactly two symmetric decompositions.*

The contact locus in the weakly defective cases.

The contact locus in the cases

- rank 9 in $S^6\mathbb{C}^3$,
- rank 8 in $S^4\mathbb{C}^4$,
- rank 9 in $S^3\mathbb{C}^6$

is an elliptic curve.
This fits exactly with the classical result that there is a *unique elliptic normal curve* passing through 9 general points in $\mathbb{P}^2$, 8 general points in $\mathbb{P}^3$, 9 general points in $\mathbb{P}^5$. The two cases in $\mathbb{P}^2$ and $\mathbb{P}^5$ correspond each other by Gale duality.

Details of the algorithm in the symmetric case

1. Pick r general vectors $a_i = \in A$.
2. Compute cartesian equations $H_1, \ldots H_e$ for the linear subspace spanned by $a_i S^{d-1} A$.
3. Compute all partial derivatives $\frac{\partial}{\partial a^i}$ of $H_s(a)$.
4. Evaluate at a_1 the Jacobian matrix of the equations got in step 3.
5. If the rank of the Jacobian in step 4 is $\dim A - 1$ then tensors in $S^d A$ are r-generically identifiable.

Perfect cases

A format $\mathbb{C}^{n_1} \otimes \ldots \otimes \mathbb{C}^{n_d}$ is called *perfect* if $1 + \sum_{i=1}^{d}(n_i - 1)$ divides $\prod_{i=1}^{d} n_i$. In perfect cases we expect finitely many decompositions of general tensors.

In the symmetric case, a format $S^d\mathbb{C}^n$ is perfect if n divides $\binom{n+d}{d}$.

Homotopic continuation method

Homotopic continuation method is quite powerful in solving $T = \sum a_i b_i c_i$ where $a_i b_i c_i$ are unknowns. The method starts from a decomposition of another tensor $T' = \sum_i a_i^1 b_i^1 c_i^1$.
We construct a path for $0 \le t \le 1$

$$(1-t)T + tT'$$

and we want to solve

$$(1-t)T + tT' = \sum_i a_i(t) b_i(t) c_i(t)$$

starting from $a_i(1) = a_i^1$, $b_i(1) = b_i^1$, $c_i(1) = c_i^1$.
Newton method is used along the path. Software `Bertini` by Bates, Hauenstein, Sommese is dedicated for this purpose.
This is usually fine in complex case, in real case it is convenient to consider $(1-t)T + t\gamma T'$ where $\gamma \in \mathbb{C}$ (the *"gamma" trick*).

Path can be a loop.

Running the method along a loop, with $T(0) = T(1)$, we may find another decomposition, starting from a given one. Surprisingly, all the decompositions may be found quickly in all the perfect cases when we know how many they are.

This gives a probabilistic guess about the number of solutions in unknown cases.

Results proved with the help of homotopic continuation method

Theorem (Hauenstein-Oeding-O-Sommese)

The general tensor of format $(3, 4, 5)$ *has a unique CP decomposition as a sum of* 6 *decomposable summands.*

Theorem (Hauenstein-Oeding-O-Sommese)

The general tensor of format $(2, 2, 2, 3)$ *has a unique CP decomposition as a sum of* 4 *decomposable summands.*

The proof uses vector bundles techniques and provide algorithms for computing the unique decomposition, which we have implemented in `Macaulay2` .

A conjecture for unsymmetric tensors

Based on the evidence described throughout, we formulate the following conjecture.

Conjecture (Hauenstein-Oeding-O-Sommese)

The only perfect formats $(n_1, \ldots, n_d)$ where a general tensor has a unique CP decomposition are:

- $(2, k, k)$ for some k — matrix pencils, known classically by Kronecker normal form
- $(3, 4, 5)$
- $(2, 2, 2, 3)$

A conjecture for symmetric tensors

In the symmetric case, the identifiable cases were known since the XIX century.

Conjecture (Mella)

The only perfect formats (n, d), where a general tensor in $S^d\mathbb{C}^n$ has a unique decomposition are:

- $(2, 2k+1)$ for some k — odd degree binary forms (Sylvester)
- $(3, 5)$ — Quintic Plane Curves (Hilbert, Richmond, Palatini)
- $(4, 3)$ — Cubic Surfaces (Sylvester Pentahedral Theorem)

General unsymmetric tensors

The following table lists all perfect "balanced" format 3-tensors with $\prod_{i=1}^{3} n_i \leq 150$.

(n_1, n_2, n_3)	gen. rank	# of decomp. of general tensor
$(3, 4, 5)$	6	**1**
$(3, 6, 7)$	9	38
$(4, 4, 6)$	8	62
$(4, 5, 7)$	10	$\geq$ 222,556

General symmetric tensors

The following table records all the known concerning $S^d\mathbb{C}^3$.

d	gen. rank	# of decomp.	reference
4	6	∞	Clebsch (1861)
5	7	1	Hilbert (1888)
6	10	∞	trivial
7	12	5	Dixon-Stuart (1906)
8	15	16	Ranestad-Schreyer (2000)
9	19	∞	trivial
10	22	320	Hauenstein-Oeding-O-Sommese (2015)
11	26	2016	Hauenstein-Oeding-O-Sommese (2015)

Note that the second column follows from Alexander-Hirschowitz Theorem.

Conclusions

- Tensors with $d \geq 3$ modes allow uniqueness of CP decomposition.
- k-generic identifiability is expected to be true for all subgeneric ranks, unless some understood exceptions. In particular it is true beyond Kruskal bound.
- The contact locus is a geometric tool which allows to detect identifiability. It can be applied with great success for generic identifiability. It can be applied with partial success for specific identifiability, due to the difficulty to check if a point on a secant variety is smooth.
- In the symmetric case, the picture for generic identifiability in subgeneric rank is complete.
- Homotopic continuation methods allow to decompose general tensors and give probabilistic guess on the number of CP decompositions in perfect cases

Thanks !!