Canonical polyadic decomposition of third-order tensors: some results on uniqueness and algebraic algorithm

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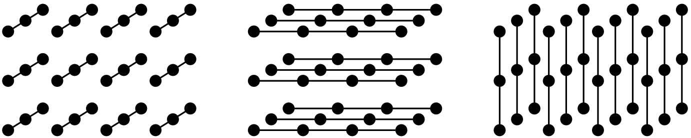
Vizualization of $\mathcal{T}$ and rank-1 tensor

$\mathcal{T} \equiv$ an $I \times J \times K$ array of numbers

rank-1 tensors

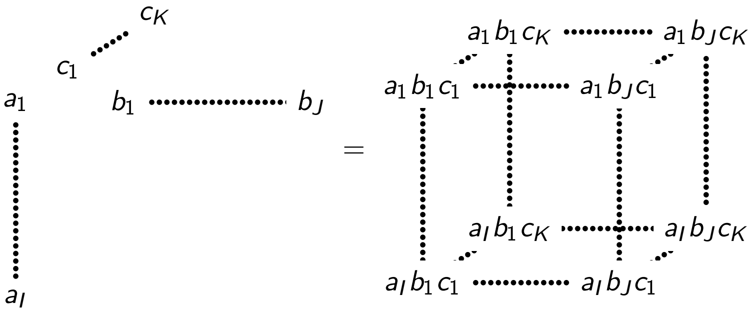


all fibers are proportional & all rows are proportional & all columns are proportional

rank-1 tensor: $\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c} \in \mathbb{C}^{I \times J \times K}$, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ - nonzero vectors

Decompositions into rank-1 terms: $I \times J \times K$ tensors

rank-1 tensor always equals the outer product of three vectors

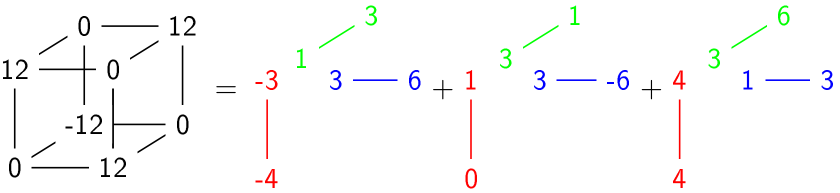


Parafac, CP decomposition,...

Polyadic Decomposition (PD): $\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r$

$$\left.\begin{array}{l}\text{Parafac or} \\ \text{Candecomp or} \\ \text{Candecomp/Parafac or} \\ \text{CP decomposition or} \\ \text{Canonical Polyadic Decomposition (CPD)}\end{array}\right\} \text{a PD with } \underbrace{\text{minimal } R}_{\text{rank of } \mathcal{T}}$$

Vizualization of PD of $\mathcal{T}$



NOTATION:

$$\mathcal{T} = \left[\underbrace{\begin{bmatrix}-3 & 1 & 4\\ -4 & 0 & 4\end{bmatrix}}_{\mathbf{A}}, \underbrace{\begin{bmatrix}3 & 3 & 1\\ 6 & -6 & 3\end{bmatrix}}_{\mathbf{B}}, \underbrace{\begin{bmatrix}1 & 3 & 3\\ 3 & 1 & 6\end{bmatrix}}_{\mathbf{C}}\right]_3 = [\underbrace{\mathbf{A}, \mathbf{B}, \mathbf{C}}_{\text{factor matrices}}]_3$$

Can we use less than 3 terms? What is the rank of $\mathcal{T}$?

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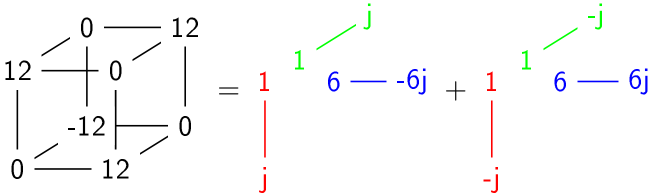


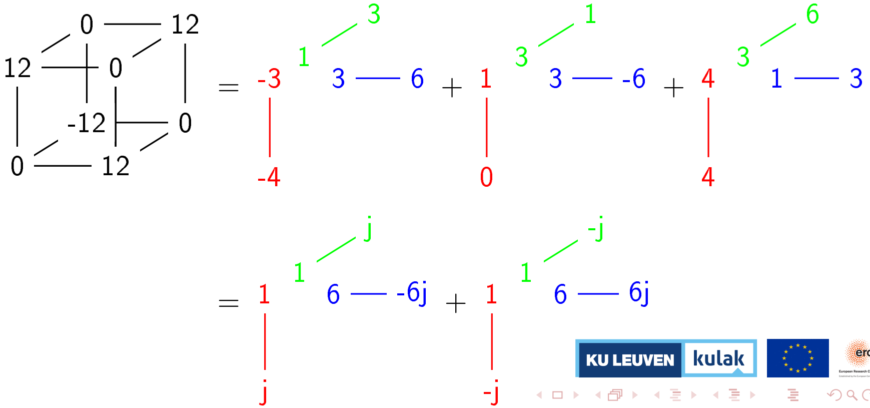
$$= \left[\!\!\left[\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}, \begin{bmatrix} 6 & 6 \\ -6j & 6j \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \right]\!\!\right]_2$$

Uniqueness of the CPD

CPD of $\mathcal{T}$ is unique

$\Updownarrow$

all CPDs of $\mathcal{T}$ have the same rank one terms



Kruskal's results on uniqueness of CPD

Definition: The k-rank of a matrix $\mathbf{A}$

$$k_{\mathbf{A}} = \max\{k : \text{ any } k \text{ columns of } \mathbf{A} \text{ are linearly independent}\}.$$

Example:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \qquad r_{\mathbf{A}} = 2, \qquad k_{\mathbf{A}} = 2$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad r_{\mathbf{B}} = 2, \qquad k_{\mathbf{B}} = 1$$

Example: $\mathbf{A}$ is generic $I \times R$ matrix $\Rightarrow \quad k_{\mathbf{A}} = \min(I, R)$

Kruskal's results on uniqueness of CPD

Theorem

Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$. *If*

$$\begin{cases} k_{\mathbf{A}} + r_{\mathbf{B}} + r_{\mathbf{C}} & \geq 2R + 2, \\ \min(r_{\mathbf{C}} + k_{\mathbf{B}}, k_{\mathbf{C}} + r_{\mathbf{B}}) & \geq R + 2, \end{cases} \tag{1}$$

or "permute **A**, **B**, **C** *in* (1)*" or*

$$k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} \geq 2R + 2, \tag{2}$$

then $r_{\mathcal{T}} = R$ *and the CPD of* $\mathcal{T}$ *is unique.*

Obviously: $k_{\mathbf{A}} \leq r_{\mathbf{A}}$, $k_{\mathbf{B}} \leq \dots \Rightarrow$ (2) is more restrictive than (1).

J.B. Kruskal. Three-way arrays: Rank and uniqueness of trilinear decompositions. *Lin. Alg. Appl.*, 18(2):95—138, 1977.

Conditions for uniqueness: Lim–Comon

Computation of k-rank is NP hard.
it is known that $\frac{1}{\mu_{\mathbf{A}}} \geq k_{\mathbf{A}}$, where

coherence of **A**: $\mu_{\mathbf{A}} := \max_{1 \leq i < j \leq R} \frac{|\mathbf{a}_i^T \mathbf{a}_j|}{\|\mathbf{a}_i\| \|\mathbf{a}_j\|}$

Idea: replace in (2) the k-ranks by coherences^{-1}

Theorem

Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$. If $\frac{1}{\mu_{\mathbf{A}}} + \frac{1}{\mu_{\mathbf{B}}} + \frac{1}{\mu_{\mathbf{C}}} \geq 2R + 2$, then $r_{\mathcal{T}} = R$ and the CPD of tensor $\mathcal{T}$ is unique.

L.-H. Lim and P. Comon. Blind multilinear identification. *IEEE Transactions on Information Theory* 60(2):1260–1280, 2014

We can also replace the k-ranks by coherences^{-1} in (1)

Theorem

Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$. *If*

$$\begin{cases} \frac{1}{\mu_{\mathbf{A}}} + r_{\mathbf{B}} + r_{\mathbf{C}} & \geq 2R + 2, \\ \min(r_{\mathbf{C}} + \frac{1}{\mu_{\mathbf{B}}}, \frac{1}{\mu_{\mathbf{C}}} + r_{\mathbf{B}}) & \geq \ R + 2, \end{cases} \tag{3}$$

then $r_{\mathcal{T}} = R$ *and the CPD of* $\mathcal{T}$ *is unique.*

REMARK: "permute **A**, **B**, **C** in (3)" – two other conditions

Problem 1:(Uniqueness)

Given CPD $\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r$. Is the CPD unique?

Problem 2:("Algebraic" algorithm)

Compute the CPD $\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r$ by means of Linear Algebra:

- computation of ranks
- computation of the null space
- eigenvalue decompositions

An $R \times R \times 2$ tensor of rank R

Given PD $\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r \in \mathbb{C}^R \times \mathbb{C}^R \times \mathbb{C}^2$,

that is, $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$, in which

$$\begin{cases} \mathbf{A} := [\mathbf{a}_1 \ \ldots \mathbf{a}_R] \text{ and } \mathbf{B} := [\mathbf{b}_1 \ \ldots \mathbf{b}_R], & \text{are } R \times R \text{ matrices,} \\ \mathbf{C} := [\mathbf{c}_1 \ \ldots \ \mathbf{c}_R] := \begin{bmatrix} \mathbf{c}^1 \\ \mathbf{c}^2 \end{bmatrix}, & \text{is an } 2 \times R \text{ matrix.} \end{cases}$$

An $R \times R \times 2$ tensor of rank R

Assume that

- $\mathbf{A} := [\mathbf{a}_1 \ \ldots \mathbf{a}_R]$ and $\mathbf{B} := [\mathbf{b}_1 \ \ldots \mathbf{b}_R]$ are nonsingular and
- $\mathbf{C}$ has no proportional columns.

Then

P1: $rk(\mathcal{T}) = R$ and the CPD of $\mathcal{T}$ is unique,

P2: CPD of $\mathcal{T}$ can be found algebraically.

An $R \times R \times 2$ tensor of rank R: reduction to ED

STEP 1: Rewrite $\mathcal{T} = \sum_{r=1}^{R} \mathbf{a}_r \otimes \mathbf{b}_r \otimes \mathbf{c}_r$ in a matrix form:

$$\mathbf{T}_1 = \mathbf{A}\mathrm{Diag}(\mathbf{c}^1)\mathbf{B}^T, \quad \mathbf{T}_2 = \mathbf{A}\mathrm{Diag}(\mathbf{c}^2)\mathbf{B}^T, \qquad (*)$$

where $\mathbf{T}_1$ and $\mathbf{T}_2$ denote the frontal slices of $\mathcal{T}$.

STEP 2: eliminate $\mathbf{A}$ and $\mathbf{B}$ in $(*)$

$$\mathbf{A}\mathrm{Diag}(\mathbf{d})\mathbf{A}^{-1} = \mathbf{T}_1\mathbf{T}_2^{-1}, \qquad \mathbf{B}\mathrm{Diag}(\mathbf{d})\mathbf{B}^{-1} = (\mathbf{T}_2^{-1}\mathbf{T}_1)^T,$$

where $\mathrm{Diag}(\mathbf{d}) := \mathrm{Diag}(\mathbf{c}^1)\mathrm{Diag}(\mathbf{c}^2)^{-1}$ has distinct diagonal entries.

STEP 3: ,

$\mathbf{a}_1, \dots, \mathbf{a}_R$ are the eigenvectors of $\mathbf{T}_1\mathbf{T}_2^{-1}$

$\mathbf{b}_1, \dots, \mathbf{b}_R$ are the eigenvectors of $(\mathbf{T}_2^{-1}\mathbf{T}_1)^T$

$\mathbf{c}^1, \mathbf{c}^2$ are easily found from $(*)$.

Definition

Let $\mathcal{C}_m(\mathbf{A}) \odot \mathcal{C}_m(\mathbf{B})$ be a matrix whose columns are

$$\Lambda^m(\mathbf{a}_{i_1} \otimes \cdots \otimes \mathbf{a}_{i_m}) \otimes \Lambda^m(\mathbf{b}_{i_1} \otimes \cdots \otimes \mathbf{b}_{i_m}), \quad 1 \leq i_1 < i_2 < \cdots < i_m \leq R.$$

Theorem (ID&LDL,2014)

Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$, $k_\mathbf{C} = r_\mathbf{C}$, $m = R - r_\mathbf{C} + 2$ *and let*

$$\mathcal{C}_m(\mathbf{A}) \odot \mathcal{C}_m(\mathbf{B}) \text{ have full column rank.}$$

Then

(i) $r_\mathcal{T} = R$ *and the CPD of* $\mathcal{T}$ *is unique;*

(ii) *the CPD of* $\mathcal{T}$ *can be found algebraically.*

I. Domanov and L. De Lathauwer. Canonical polyadic decomposition of third-order tensors: reduction to generalized eigenvalue decomposition. *SIAM J. Matrix Anal. Appl.*, 35(2):636-660, 2014.

Illustration for $K = R$: uniqueness

We re-explain results from

L. De Lathauwer. A link between the canonical decomposition in multilinear algebra and simultaneous matrix diagonalization. *SIAM J. Matrix Anal. Appl.*, 28(3): 642-666, 2006.

T. Jiang and N.D. Sidiropoulos. Kruskal's permutation lemma and the identification of CANDECOMP/PARAFAC and bilinear models with constant modulus constraints. *IEEE Trans. Signal Process.*, 52(9):2625-2636, 2004.

$K = R \;\Rightarrow m = R - r_{\mathbf{C}} + 2 = 2$

$\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B})$ has $\frac{R(R-1)}{2}$ columns

$$(\mathbf{a}_i \otimes \mathbf{a}_j - \mathbf{a}_j \otimes \mathbf{a}_i) \otimes (\mathbf{b}_i \otimes \mathbf{b}_j - \mathbf{b}_j \otimes \mathbf{b}_i), \quad 1 \leq i < j \leq R$$

If $\mathcal{C}_2(\mathbf{A}) \odot \mathcal{C}_2(\mathbf{B})$ has f.c.r $\Rightarrow$ CPD is unique

OLD: Illustration for $K = R$: algebraic algorithm

Main construction:

INPUT: $\mathcal{T} \in \mathbb{C}^{I \times J \times K}$
OUTPUT: $\mathbf{R}_2(\mathcal{T}) \in \mathbb{C}^{I^2 J^2 \times K^2}$

STEP 1: $\mathcal{T}^{\otimes^2} \leftarrow \mathcal{T} \otimes \mathcal{T} \in \mathbb{C}^{I \times J \times K \times I \times J \times K}$
STEP 2: $\mathcal{T}_I^{\otimes^2} \leftarrow$ partially skew-symmetrize $\mathcal{T}^{\otimes^2}$ along "I" dim-s
STEP 3: $\mathcal{T}_{IJ}^{\otimes^2} \leftarrow$ partially skew-symmetrize $\mathcal{T}_I^{\otimes^2}$ along "J" dim-s
STEP 4: $\mathbf{R}_2(\mathcal{T}) \leftarrow$ reshape $\mathcal{T}_{IJ}^{\otimes^2}$ into $I^2 J^2 \times K^2$ matrix

OLD: Illustration for $K = R$: algebraic algorithm

Properties of $\mathbf{R}_2(\mathcal{T})$

- rows of $\mathbf{R}_2(\mathcal{T})$ are vectorized $R \times R$ symmetric matrices
- $\dim \underbrace{\left(\ker(\mathbf{R}_2(\mathcal{T})) \bigcap (\text{ vectorized symmetric matrices})\right)}_{W} = R$
- $\mathbf{R}_2(\mathcal{T})(\mathbf{x} \otimes \mathbf{x}) = \mathbf{0} \Leftrightarrow \mathbf{x}$ is proportional to a column of $\mathbf{C}^{-T}$

$\Rightarrow$

if $\mathbf{w}_1, \ldots, \mathbf{w}_R$ is a basis of W,
then there exists a unique nonsingular $R \times R$ matrix $\mathbf{M}$:

$[\mathbf{w}_1 \ \ldots \ \mathbf{w}_R] = (\mathbf{C}^{-T} \odot \mathbf{C}^{-T})\mathbf{M} \sim$ CPD of $R \times R \times R$ tensor of rank R

$$\mathcal{W} = [\mathbf{C}^{-T}, \mathbf{C}^{-T}, \mathbf{M}^T]_R$$

NEW: Algorithm for $K = R$

I. Domanov, L. De Lathauwer. Canonical polyadic decomposition of third-order tensors: relaxed uniqueness conditions and algebraic algorithm, arXiv:1501.07251, 2015.

INPUT: $\mathcal{T} \in \mathbb{C}^{I\times J\times R}$ and $l \geq 0$
OUTPUT: Matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ such that $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$

1: Construct the $I^{2+l}J^{2+l} \times R^{2+l}$ matrix $\mathbf{R}_{2,l}(\mathcal{T})$
2: Find a basis $\mathbf{w}_1, \ldots, \mathbf{w}_R$ of $\ker(\mathbf{R}_{2,l}(\mathcal{T})) \bigcap S^{2+l}(\mathbb{R}^{R^{2+l}})$
3: $\mathbf{W} \leftarrow [\mathbf{w}_1 \ \ldots \ \mathbf{w}_R]$
4: Reshape the $R^{2+l} \times R$ matrix $\mathbf{W}$ into an $R \times R^{1+l} \times R$ tensor $\mathcal{W}$
5: Compute the CPD $\mathcal{W} = [\mathbf{C}^{-T}, \mathbf{C}^{-T} \odot \cdots \odot \mathbf{C}^{-T}, \mathbf{M}]_R$
6: Find the columns of $\mathbf{A}$ and $\mathbf{B}$ (trivial)

Example: $I \times J \times K$ tensor of rank $R = K = (I-1)(J-1)$

$I \times J \times (I-1)(J-1)$	l	C_{R+l+1}^{2+l}	t_1(sec)	t_2(sec)
$3 \times 3 \times 4$	0	10	0.012	0.008
$3 \times 4 \times 6$	0	21	0.022	0.013
$3 \times 5 \times 8$	0	36	0.038	0.013
$3 \times 6 \times 10$	0	55	0.060	0.014
$3 \times 7 \times 12$	1	364	0.368	0.035
$3 \times 8 \times 14$	1	560	0.725	0.071
$3 \times 9 \times 16$	1	816	1.342	0.156
$3 \times 10 \times 18$	1	1140	2.333	0.284
$3 \times 11 \times 20$	1	1540	4.259	0.773
$3 \times 12 \times 22$	1	2024	6.119	0.970
$3 \times 13 \times 24$	1	2600	9.386	1.698
$4 \times 4 \times 9$	0	45	0.047	0,013
$4 \times 5 \times 12$	1	364	0.367	0.034
$4 \times 6 \times 15$	1	680	0.988	0.108
$4 \times 7 \times 18$	2	5985	22.375	8.566
$4 \times 8 \times 21$	2	10626	56.758	36.272
$4 \times 9 \times 24$	2	17550	150.261	210.018
$5 \times 5 \times 16$	1	816	1.321	0.152
$5 \times 6 \times 20$	2	8855	41.213	22.903
$5 \times 7 \times 24$	2	17550	139.622	212.346
$6 \times 6 \times 25$	2	20475	771.171	443.346

Comparision with Tensorlab: $3 \times 7 \times 12$ tensor of rank 12

$\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_{12}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 & 0 & 6 & 6 & 7 & 9 & 0 & 8 \\ 2 & 3 & 5 & 7 & 0 & 6 & 6 & 7 & 9 & 0 & 8 & 2 \\ 3 & 5 & 7 & 0 & 6 & 6 & 7 & 9 & 0 & 8 & 2 & 1 \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 4 & 5 & 6 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 5 & 6 & 7 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 6 & 7 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 0 & 1 & 2 & 3 \end{bmatrix}, \qquad \mathbf{C} = \mathbf{I}_{12}$$

Our algorithm: takes less than 1 second
Tensorlab: cannot find with 500 random initializations
(Gauss-Newton dogleg trust region method, max_iter =10000)

Conjecture

Our algorithm can compute the CPD of a generic $I \times J \times K$ tensor of rank $R \leq K \leq (I-1)(J-1)$.

confirmed numerically for $R \leq 27$.

Case $K \leq R$: similar results on uniqueness and algorithm

$\mathbf{R}_{m,l}(\mathcal{T})$ is constructed for $m = R - K + 2 \geq 2$

	$I \times J \times K$	R	m	l	C_{R+l+m}^{m+l-1}	t_1(sec)	t_2(sec)
1	$4 \times 5 \times 6$	7	3	1	126	0.139	0.043
2	$5 \times 7 \times 7$	9	4	1	462	1.352	0.246
3	$6 \times 9 \times 8$	11	5	1	1716	18.969	9.647
4	$7 \times 7 \times 7$	10	5	1	924	6.227	1.965
5	$4 \times 6 \times 8$	9	3	1	330	0.553	0.077
6	$4 \times 7 \times 10$	11	3	1	715	1. 996	0.356
7	$5 \times 6 \times 6$	8	4	2	462	1.091	0.165
8	$5 \times 7 \times 8$	10	4	2	1716	11.396	3.156
9	$6 \times 7 \times 6$	9	5	3	1287	29.426	2.054

Corollary

Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$. *Suppose that*

$$k_{\mathbf{A}} + r_{\mathbf{B}} + k_{\mathbf{C}} \geq 2R + 2, \quad \text{and} \quad k_{\mathbf{B}} + k_{\mathbf{C}} \geq R + 2.$$

Then $r_{\mathcal{T}} = R$ *and the CPD of tensor* $\mathcal{T}$ *is unique and can be found algebraically.*

Corollary

Let $\mathcal{T} = [\mathbf{A}, \mathbf{B}, \mathbf{C}]_R$ *and let* $k_{\mathbf{A}} + k_{\mathbf{B}} + k_{\mathbf{C}} \geq 2R + 2$.
Then the CPD of $\mathcal{T}$ *is unique and can be found algebraically.*

Future work

- Higher-order tensor with a long dimension
- Perturbation theory

Thank you for your attention.