Candecomp/Parafac based Array Processing

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Winter school : Search for Latent Variables : ICA, Tensors and NMF

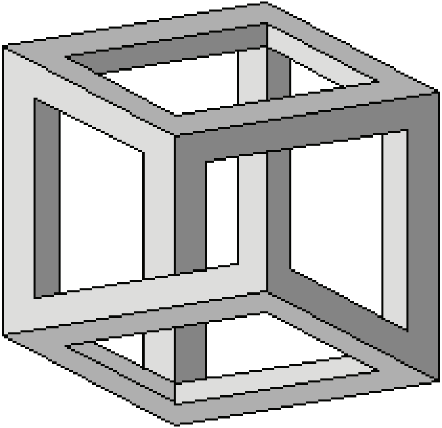
2 – 4 February 2015, Villard de Lans

Candecomp/Parafac Decompositions

CP based Array Processing

CP-Based Vector Sensor Array Processing

Candecomp/Parafac Decompositions



Useful matrix operations

$$\mathbf{A} = [\mathbf{a}_1 \cdots \mathbf{a}_R],\ (I \times R) \quad \mathbf{B} = [\mathbf{b}_1 \cdots \mathbf{b}_R],\ (J \times R)$$

– Kronnecker product

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{A}(1,1)\mathbf{B} & \mathbf{A}(1,2)\mathbf{B} & \cdots \\ \mathbf{A}(2,1)\mathbf{B} & \mathbf{A}(2,2)\mathbf{B} & \cdots \\ & \vdots & \end{bmatrix},\ (IJ \times R^2)$$

– Khatri-Rao product

$$\mathbf{A} \odot \mathbf{B} = [\mathbf{a}_1 \otimes \mathbf{b}_1 \cdots \mathbf{a}_R \otimes \mathbf{b}_R],\ (IJ \times R)$$

– Outer product

$$\mathbf{a} \circ \mathbf{b},\ (I \times J), \quad \mathbf{a} \circ \mathbf{b}(i,j) = \mathbf{a}(i)\mathbf{b}(j) \text{ \textit{i.e.} } \mathbf{a} \circ \mathbf{b} = \mathbf{a}\mathbf{b}^T,\ \text{rank 1 matrix.}$$

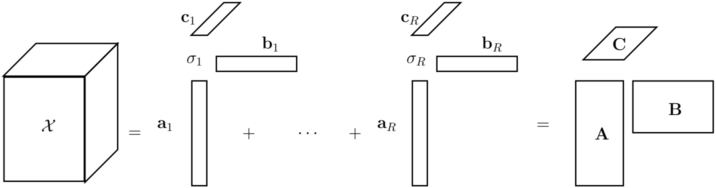
$$\mathbf{a} \circ \mathbf{b} \circ \mathbf{c},\ (I \times J \times K), \quad \mathbf{a} \circ \mathbf{b} \circ \mathbf{c}(i,j,k) = \mathbf{a}(i)\mathbf{b}(j)\mathbf{c}(k) \text{ \textit{i.e.} rank 1 tensor.}$$

CP Decomposition of an order 3 tensor

CP : Candecomp/Parafac [Harshman, 1970-1972], [Carroll - Chang, 1970]

$$\mathcal{X} \approx \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r = [\![\mathbf{A}, \mathbf{B}, \mathbf{C}]\!]$$

$$\mathbf{A}(I \times R),\ \mathbf{B}(J \times R),\ \mathbf{C}(K \times R)$$



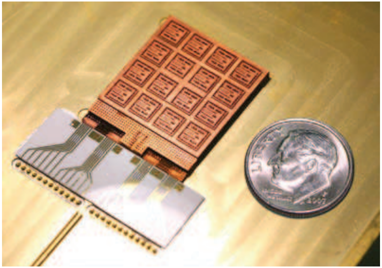
Tensor rank : minimal number of rank 1 tensors to represent $\mathcal{X}$.

Alternative writing

Slice : $\mathbf{X}_k = \mathbf{A}\mathbf{D}_k(\mathbf{C})\mathbf{B}^T, \quad k = 1, \cdots, K$

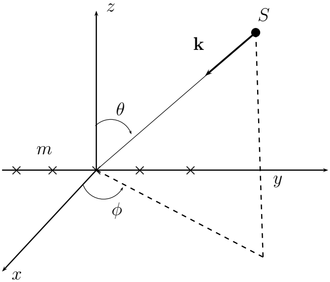
Unfolding : $\mathbf{X}^{(KJ \times I)} = (\mathbf{B} \odot \mathbf{C})\mathbf{A}^T$

CP based Array Processing



Antenna arrays

Direction Of Arrival (DOA)



Far field EM wave $\Rightarrow$ **plane wave**

Direction cosine $\mathbf{k}$

$$\mathbf{k} = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}$$

Narrowband assumption

Received signal on sensor 1 : $s(t)\mathrm{e}^{j\frac{2\pi}{\lambda}t}$

Received signal on sensor m : $s(t-t_m)\mathrm{e}^{j\frac{2\pi}{\lambda}(t-t_m)} \approx \underbrace{s(t)\mathrm{e}^{j\frac{2\pi}{\lambda}t_m}}_{\text{complex envelope}} \underbrace{\mathrm{e}^{j\frac{2\pi}{\lambda}t}}_{\text{carrier}}$

Phase factor : $\mathrm{e}^{j\frac{2\pi}{\lambda}t_m} = \mathrm{e}^{j\frac{2\pi}{\lambda}\mathbf{k}^T\mathbf{d}_m}$ where $\mathbf{d}_m$ is the position of the m^{th} sensor

Multiple Sources

P narrowband sources

Steering vector of the p^{th} source : $\mathbf{a}(\mathbf{k}_p) = [\mathrm{e}^{j\frac{2\pi}{\lambda}\mathbf{k}^T\mathbf{d}_1} \cdots \mathrm{e}^{j\frac{2\pi}{\lambda}\mathbf{k}^T\mathbf{d}_M}]^T$

$$\mathbf{X} = \mathbf{A}\mathbf{S}^T + \mathbf{N} \quad \dim(\mathbf{X}) = (M \times K)$$

$$\begin{aligned} \mathbf{A} &= \left[\mathbf{a}(\mathbf{k}_1), \ldots, \mathbf{a}(\mathbf{k}_P)\right] \ (M \times P) \\ \mathbf{S} &= \left[\mathbf{s}_1,\ \mathbf{s}_2, \ldots, \mathbf{s}_P\right] \ (K \times P) \end{aligned}$$

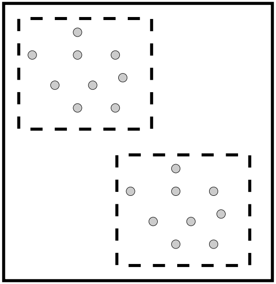
DOA estimation $\Rightarrow$ estimation of $\mathbf{A}$ $\Rightarrow$ estimation of $\mathbf{k}_p, p = 1, \cdots, P$

Unambiguous estimation $\Rightarrow$ inter-element spacing $\leq \lambda/2$

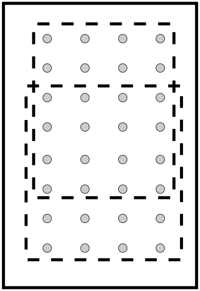
Eigen decomposition approaches (Music, Esprit)

Rotational Invariance

Basic idea of Esprit : the array can be decomposed into 2 translated but otherwise identical subarrays ⇒ GEVD of stacked data corresponding to each subarray



Two non-overlapping subarrays



Two overlapping subarrays

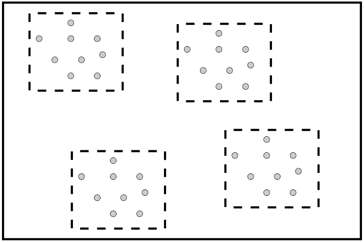
The idea of Esprit is difficult to extend to multiple invariances !

CP-based Array Processing

Seminal work of Sidiropoulos, Bro and Giannakis (2000)

$\Rightarrow$ Handling multiple invariances

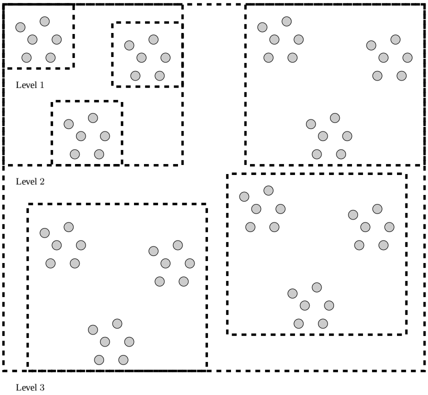
$\Rightarrow$ Joint estimation of both DOA and sources



An array with multiple invariances

$$\mathbf{X} = \big(\mathbf{A}_1 \odot \mathbf{A}_2\big)\mathbf{S}^T + \mathbf{N}$$

Multi-scale Array



$$\mathbf{d}_{l_1,l_2,\ldots,l_N} = \sum_{n=1}^{N} \mathbf{d}_{l_n}^{n}$$

Data Model (1)

Phase factor for one narrowband source :

$$a_{l_1,l_2,\ldots,l_N}(\mathbf{k}) = \exp\left\{j\frac{2\pi}{\lambda}\ \sum_{n=1}^{N}\mathbf{k}^T\mathbf{d}_{l_n}^n\right\} = \prod_{n=1}^{N}\exp\left\{j\frac{2\pi}{\lambda}\ \mathbf{k}^T\mathbf{d}_{l_n}^n\right\}.$$

Array manifold for the entire sensor array

$$\mathbf{a}(\mathbf{k}) = \mathbf{a}_1(\mathbf{k}) \otimes \cdots \otimes \mathbf{a}_N(\mathbf{k}),$$

with

$$\mathbf{a}_n(\mathbf{k}) = \begin{bmatrix} e^{j(2\pi/\lambda)\mathbf{k}^T\mathbf{d}_1^n} \\ \vdots \\ e^{j(2\pi/\lambda)\mathbf{k}^T\mathbf{d}_{L_n}^n} \end{bmatrix}$$

$\otimes$ Kronecker product

Data Model (2)

P narrowband sources

$$\begin{aligned}
\mathbf{A}_1 &= \left[\mathbf{a}_1(\mathbf{k}_1), \ldots, \mathbf{a}_1(\mathbf{k}_P)\right] \ (L_1 \times P) \\
&\vdots \\
\mathbf{A}_N &= \left[\mathbf{a}_N(\mathbf{k}_1), \ldots, \mathbf{a}_N(\mathbf{k}_P)\right] \ (L_N \times P) \\
\mathbf{S} &= \left[\mathbf{s}_1,\ \mathbf{s}_2, \ldots, \mathbf{s}_P\right] \ (K \times P)
\end{aligned}$$

$$\mathbf{X} = \big(\mathbf{A}_1 \odot \cdots \odot \mathbf{A}_N\big)\mathbf{S}^T + \mathbf{N},$$

$\Rightarrow$ **CP Model of order** $N+1$

Single snapshot $\Rightarrow$ **CP Model of order** N

Model identifiability

- [Sidiropoulos and Bro 2000] : sufficient condition for the uniqueness of CP decomposition for N-way arrays
- P sources with distinct DOAs and not fully correlated
- Number of snapshots greater than number of sources ($K > P$)

$$\sum_{n=1}^{N} \min(L_n, P) \geq P + N.$$

- Case of a single snapshot

$$\sum_{n=1}^{N} \min(L_n, P) \geq 2P + N - 1.$$

Parameter estimation (1)

Estimation of the steering vectors by CP decomposition of the data

$\hat{\mathbf{a}}_n^p$: nth level estimated steering vector for the pth source

Estimating the DOA parameters for the pth source $\Rightarrow$ minimization of the following criterion :

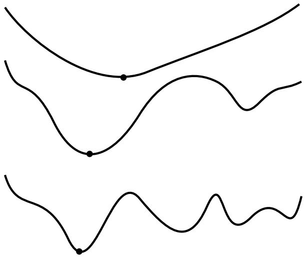
$$\mathcal{I}_N(\mathbf{k}_p) = \sum_{n=1}^{N} \mathcal{J}_n(\mathbf{k}_p).$$

$$\mathcal{J}_n(\mathbf{k}_p) = \|\hat{\mathbf{a}}_n^p - \mathbf{a}_n(\mathbf{k}_p)\|^2, \;\; n = 1, \ldots, N,$$

Parameter estimation (2)

$\mathcal{I}_N(\mathbf{k}_p)$: non-convex criterion highly non-linear

$\Rightarrow$ Sequential strategy similar to a GNC approach



$\mathcal{I}_1(\mathbf{k}_p),\ \Delta \leq \lambda/2$

$\mathcal{I}_2(\mathbf{k}_p)$

$\mathcal{I}_3(\mathbf{k}_p)$

$\Rightarrow$ The ordering of the sub-arrays is important !

Parameter estimation (3)

First Stage : Estimate $\mathbf{A}_1, \ldots, \mathbf{A}_N$ by CP decomposition of the data.

Second Stage :

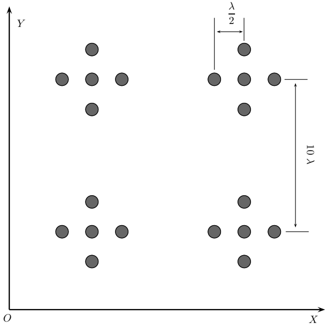
- For $p = 1, \ldots, P$ and

  for $n = 1, \ldots, N$ compute

$$\mathbf{k}^*_{p,n} = \underset{\mathbf{k}_p}{\operatorname{argmin}}\, \mathcal{I}_n(\mathbf{k}_p).$$

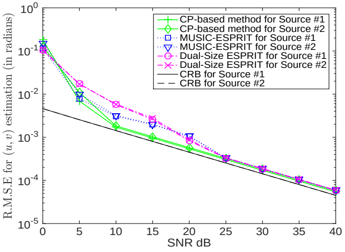
- *Output :* The estimated parameters for the P sources : $\hat{\mathbf{k}}_p = (\hat{u}_p, \hat{v}_p, \hat{w}_p) = \mathbf{k}^*_{p,N}$ with $p = 1, \ldots, P$.

Simulation results (1)

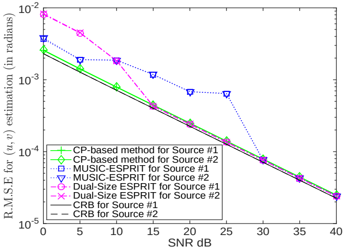


- Two scales $L_1 = 5, L_2 = 4$
- Three scales $L_1 = 5, L_2 = 2, L_3 = 2 \Rightarrow$ single snapshot case
- Two sources with distinct DOAs
- Comparison with ESPRIT [Wong, Zoltowski 1998]

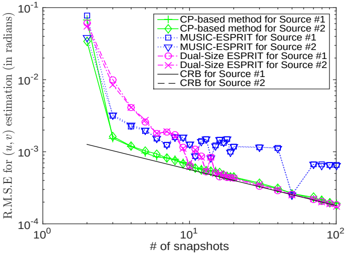
Simulation results (2)



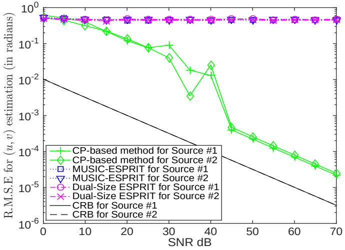
CRMSE , K= 5



CRMSE , K=20

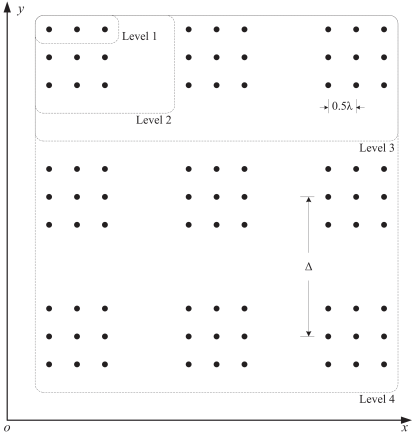


CRMSE , SNR $= 15$ dB



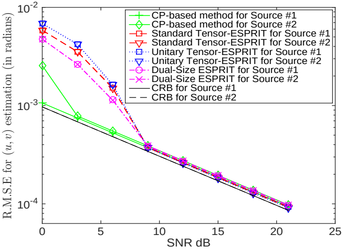
CRMSE , K=1

Simulation results (3)

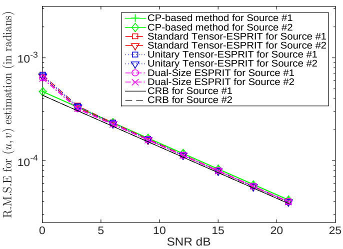


- Four scales $L_1 = L_2 = L_3 = L_4 = 3$
- Two sources with distinct DOAs
- Comparison with ESPRIT [Wong, Zoltowski 1998] and Tensor-ESPRIT [Haardt et al. 2008]

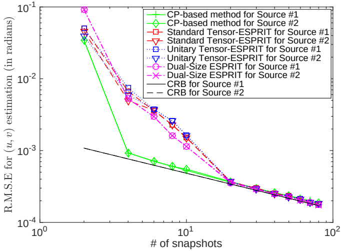
Simulation results (4)



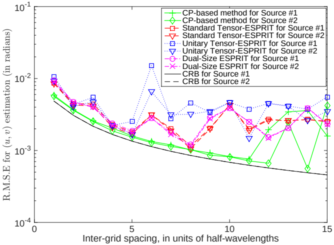
CRMSE , K= 10



CRMSE , K=50



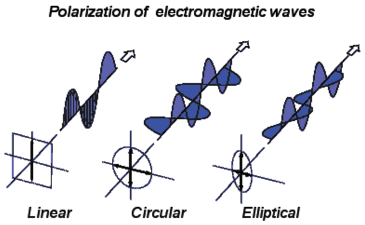
CRMSE , SNR = 6 dB

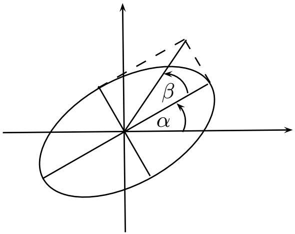


CRMSE , K=5, SNR = 6 dB

CP-Based Vector Sensor Array Processing

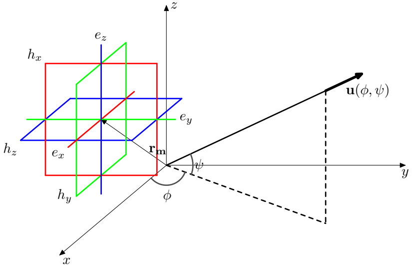
Polarized EM wave





α : orientation β : ellipciticity

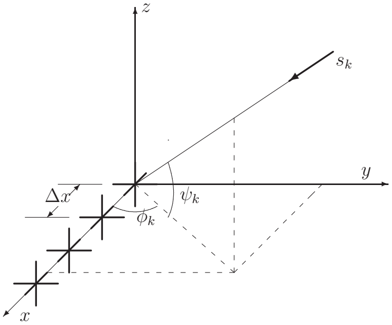
Response of the vector sensor to a polarized wave



$$\mathbf{g}_k \triangleq \mathbf{g}(\alpha_k, \beta_k) = \begin{bmatrix} g_\phi(\alpha_k, \beta_k) \\ g_\psi(\alpha_k, \beta_k) \end{bmatrix} = \begin{bmatrix} \cos\alpha_k & \sin\alpha_k \\ -\sin\alpha_k & \cos\alpha_k \end{bmatrix} \begin{bmatrix} \cos\beta_k \\ j\sin\beta_k \end{bmatrix}$$

$$\mathbf{b}_k \triangleq \begin{bmatrix} \mathbf{e}(\phi_k, \psi_k, \alpha_k, \beta_k) \\ \mathbf{h}(\phi_k, \psi_k, \alpha_k, \beta_k) \end{bmatrix} = \underbrace{\begin{bmatrix} -\sin\phi_k & -\cos\phi_k \sin\psi_k \\ \cos\phi_k & -\sin\phi_k \sin\psi_k \\ 0 & \cos\psi_k \\ -\cos\phi_k \sin\psi_k & \sin\phi_k \\ -\sin\phi_k \sin\psi_k & -\cos\phi_k \\ \cos\psi_k & 0 \end{bmatrix}}_{\mathbf{F}(\phi_k, \psi_k)} \mathbf{g}_k.$$

CP model of a Vector Sensor Array



$$\mathbf{X} = (\mathbf{A} \odot \mathbf{B})\mathbf{S}^T + \mathbf{N}$$

$$\mathbf{A} = [\mathbf{a}_1(\phi_1, \psi_1), \ldots, \mathbf{a}_K(\phi_K, \psi_K)]$$

$(M \times K)$ steering matrix of the K sources

$$\mathbf{B} = [\mathbf{b}_1(\phi_1, \psi_1, \alpha_1, \beta_1), \ldots, \mathbf{b}_K(\phi_K, \psi_K, \alpha_K, \beta_K)]$$

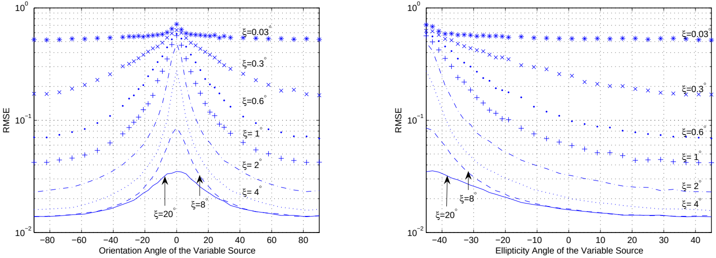
$(6 \times K)$ polarization matrix of the K sources

Does polarization really matter ?

Two key factors directly affect the performance of polarized source estimation :

– the polarization separation

– the angular separation



(a) $\beta_1 = \beta_2 = 0^\circ$, $\alpha_1 = 0^\circ$ varying α_2 (b) $\alpha_1 = \alpha_2 = 0^\circ$, $\beta_1 = -45^\circ$ varying β_2

FIGURE: The RMSE of source DOA estimation versus their polarization separation