Input-to-State Stability
of Nonlinear Functional Systems

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Outline

- ISS, ISS-ation for Delay-Free Systems
- ISS for Systems Described by RFDEs,
- ISS for Systems Described by FDEs
- ISS for Systems Described by NFDEs
- ISS-ation of Systems Described by RFDEs
- A Case Study: the Chemical Reactor with Recycle
- Conclusions
A function $\delta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is:

- positive definite if it is continuous, zero at zero and $\delta(s) > 0$ for all $s > 0$ (ex: $s \rightarrow \frac{s}{1+s^2}$);

- of class $\mathcal{K}$ if it is positive definite and strictly increasing (ex: $s \rightarrow 1 - e^{-s}$);

- of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded (ex: $s \rightarrow s^2$);

- of class $\mathcal{L}$ if it is continuous and it monotonically decreases to zero as its argument tends to $+\infty$ (ex: $s \rightarrow e^{-s}$).

A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is of class $\mathcal{K}\mathcal{L}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \geq 0$ (ex: $(s, t) \rightarrow se^{-t}$).
For positive real $\Delta$, positive integer $n$, $C([-\Delta, 0]; R^n)$ denotes the Banach space of the continuous functions mapping $[-\Delta, 0]$ into $R^n$, endowed with the supremum norm, denoted with the symbol $\| \cdot \|_\infty$.

The symbol $\| \cdot \|_a$ denotes any semi-norm in $C([-\Delta, 0]; R^n)$ for which there exist two positive reals $\gamma_a$ and $\overline{\gamma}_a$ such that, for any $\phi \in C([-\Delta, 0]; R^n)$, the following inequalities hold

$$\gamma_a | \phi(0) | \leq \| \phi \|_a \leq \overline{\gamma}_a \| \phi \|_\infty$$

The same symbol $\| \cdot \|_a$ denotes any semi-norm in $R^n \times C([-\Delta, 0]; R^n)$ for which there exist two positive reals $\gamma_a$ and $\overline{\gamma}_a$ such that, for any $x \in R^n$ and any $\phi \in C([-\Delta, 0]; R^n)$, the following inequalities hold

$$\gamma_a | x | \leq \left\| \begin{bmatrix} x \\ \phi \end{bmatrix} \right\|_a \leq \overline{\gamma}_a (| x | + \| \phi \|_\infty)$$
A functional $V : C([−\Delta,0]; R^n) \to R^+$ is Fréchet differentiable at a point $φ \in C([−\Delta,0]; R^n) \to R^+$, if there exists a linear bounded operator, which is called the Fréchet differential at $φ$ and is denoted as $D_F V(φ)$, mapping $C([−\Delta,0]; R^n)$ into $C([−\Delta,0]; R^n)$, such that

$$\lim_{ψ \to 0} \frac{|V(φ + ψ) - V(φ) - D_F V(φ)ψ|}{∥ψ∥_\infty} = 0$$

In the following:

- RFDE stands for Retarded Functional Differential Equation.
- NFDE stands for Neutral Functional Differential Equation.
- FDE stands for Functional Difference Equation.
- ISS stands for Input-to-State Stability, or Input-to-State Stable.
ISS Definition (Sontag, 1989)

\[
\dot{x}(t) = f(x(t), v(t)), \text{ a.e.} \quad x(t) \in R^n, \; v(t) \in R^m, \quad x(0) = x_0
\]

(1)

(\(f\) continuously differentiable)

**Definition 1.** The system described by (1) is ISS if there exist \(\beta \in \mathcal{KL}\) and \(\gamma \in \mathcal{K}\) such that, for any initial state \(x_0\) and any Lebesgue measurable and locally essentially bounded input \(v\), the solution exists for all \(t \geq 0\) and, furthermore, satisfies the inequality

\[
|x(t)| \leq \beta(|x_0|, t) + \gamma(\|v_{[0,t]}\|_{\infty}), \quad t \geq 0
\]
Liapunov Characterization, Sontag, 1989

**Theorem 2.** The system described by the ODE (1) is ISS if and only if there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}^+$, functions $\alpha_1$, $\alpha_2$ of class $\mathcal{K}_\infty$, functions $\alpha_3$, $\rho$ of class $\mathcal{K}$, such that

$$H_1) \quad \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \ \forall x \in \mathbb{R}^n;$$

$$H_2) \quad \frac{\partial V(x)}{\partial x} f(x, v) \leq -\alpha_3(|x|), \ \forall x \in \mathbb{R}^n, v \in \mathbb{R}^m : |x| \geq \rho(v).$$
ISS-ation (Sontag, 1989)

\[
\dot{x}(t) = f(x(t)) + g(x(t))(u(t) + d(t))
\]

Hp) \( u(t) = k(x(t)) \) is stabilizing when \( d \equiv 0 \), \( V : R^n \rightarrow R^+ \) is a Liapunov function for \( \dot{x}(t) = f(x(t)) + g(x(t))k(x(t)) \), i.e.: \( \alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \ \frac{\partial V(x)}{\partial x}(f(x) + g(x)k(x)) \leq -\alpha_3(|x|) \);

Th) \( u_s(t) = k(x(t)) - \left[ \frac{\partial V(x(t))}{\partial x(t)} g(x(t)) \right]^T \) is ISS-ing, i.e.

\[
\dot{x}(t) = f(x(t)) + g(x(t))(u_s(t) + d(t))
\]

is ISS w.r.t. the disturbance \( d(t) \).
Example for ISS-ation, Sontag, 1989

\[ \dot{x}(t) = x(t) + (1 + x^2(t))(u(t) + d(t)) \]

If \( d(t) \equiv 0 \), then \( u(t) = - \frac{2x(t)}{1+x^2(t)} \) is a stabilizing feedback control law. Indeed, the closed-loop system becomes \( \dot{x}(t) = -x(t) \).

But, with this feedback control law, the closed-loop system is described, in the case \( d(t) \neq 0 \), by the equation

\[ \dot{x}(t) = -x(t) + (1 + x^2(t))d(t), \]

and it can easy become unstable, for instance by suitable constant disturbance \( d(t) \).
Now, we consider a Liapunov function for the disturbance-free closed loop system $\dot{x}(t) = -x(t)$. We can choose $V(x) = x^2$. Then we have the new feedback control law

$$u_s(t) = -\frac{2x(t)}{1 + x^2(t)} - 2x(t) \left( 1 + x^2(t) \right)$$

The new closed-loop system becomes

$$\dot{x}(t) = -x(t) - 2x(t) \left( 1 + x^2(t) \right)^3 + \left( 1 + x^2(t) \right) d(t)$$

This system is ISS w.r.t. the disturbance $d(t)$. 
Scheme of a Continuous Stirred Tank Reactor.

Delays appear because of the recycle.
Human Glucose-Insulin System. Delays occur because of the reaction time of the pancreas to plasma-glucose variations.
The beginning of ISS for time-delay systems


Systems Described by RFDEs

\[ \dot{x}(t) = f(x_t, v(t)), \quad t \geq 0, \quad a.e., \]
\[ x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \]

(2)

\[ f : C([-\Delta, 0]; \mathbb{R}^n) \times \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ Lipschitz on bounded sets,} \]

\[ x_t \in C([-\Delta, 0]; \mathbb{R}^n), \quad x_t(\tau) = x(t + \tau), \quad \tau \in [-\Delta, 0] \]
An example (recall $x_t(\tau) = x(t + \tau)$, $\tau \in [-\Delta, 0]$):

$$\dot{x}(t) = x^4(t) + x^3(t-\pi) + x^2(t-e) + x(t - \sqrt{3}) + \int_{t-\sqrt{2}}^t x^5(s)ds + v(t)$$

(3)

Setting $\Delta = \pi$ (maximum involved time delay), by equalities

$x(t) = x_t(0)$, $x(t-\pi) = x_t(-\pi)$, $x(t-e) = x_t(-e)$, $x(t - \sqrt{3}) = x_t(-\sqrt{3})$,

$$\int_{t-\sqrt{2}}^t x^5(s)ds = \int_{-\sqrt{2}}^0 x^5(t + \tau)d\tau = \int_{-\sqrt{2}}^0 x_t^5(\tau)d\tau,$$

the system described by (3) can be rewritten in the form

$\dot{x}(t) = f(x_t, v(t))$, where $f : C([-\Delta, 0]; R) \times R \rightarrow R$ is defined, for $\phi \in C([-\Delta, 0]; R)$, $u \in R$, as

$$f(\phi, u) = \phi^4(0) + \phi^3(-\pi) + \phi^2(-e) + \phi(-\sqrt{3}) + \int_{-\sqrt{2}}^0 \phi^5(s)ds + u$$
Existence and Uniqueness of the Solution

**Theorem 3.** For any initial condition $x_0 \in C([-\Delta, 0]; R^n)$ and any Lebesgue measurable and locally essentially bounded input function $u$, the RFDE (2) admits a unique, locally absolutely continuous, solution $x(t)$ on a maximal time interval $[0, b)$, $0 < b \leq +\infty$. If $b < +\infty$, then the solution is unbounded in $[0, b)$. 
Stability Definitions

Definition 4. Let in the RFDE (2) $u(t) \equiv 0$. The system described by the RFDE (2) is said to be $0$–GAS if there exist a function $\beta$ of class $KL$ such that, for any $x_0 \in C([-\Delta, 0] ; \mathbb{R}^n)$, the corresponding solution exists for all $t \geq 0$ and, furthermore, satisfies the inequality

$$|x(t)| \leq \beta(\|x_0\|_\infty, t), \quad \forall t \geq 0 \quad (4)$$

Definition 5. The system described by the RFDE (2) is said to be ISS if there exist a function $\beta$ of class $KL$ and a function $\gamma$ of class $K$ such that, for any initial condition $x_0 \in C([-\Delta, 0] ; \mathbb{R}^n)$ and any Lebesgue measurable, locally essentially bounded input $v$, the corresponding solution exists for all $t \geq 0$ and, furthermore, satisfies

$$|x(t)| \leq \beta(\|x_0\|_\infty, t) + \gamma(\|v_{[0,t]}\|_\infty), \quad \forall t \geq 0.$$
Definition 6. Let \( V : C([-\Delta, 0]; R^n) \to R^+ \) be a continuous functional. The derivative \( D^+ V : C([-\Delta, 0]; R^n) \times R^m \to R^* \) of the functional \( V \) is defined, in the Driver’s form (see Driver, 1962, Burton, 1985, Pepe & Jiang, 2006, Karafyllis, 2006), for \( \phi \in C([-\Delta, 0]; R^n), \ v \in R^m \), as follows

\[
D^+ V(\phi, v) = \limsup_{h \to 0^+} \frac{1}{h} \left( V(\phi_{h,v}) - V(\phi) \right),
\]

where \( \phi_{h,v} \in C([-\Delta, 0]; R^n) \) is given by

\[
\phi_{h,v}(s) = \begin{cases} 
\phi(s + h), & s \in [-\Delta, -h], \\
\phi(0) + f(\phi, v)(h + s), & s \in (-h, 0]
\end{cases}
\]
**Theorem 7.** Let in the RFDE (2) $u(t) = 0$, $t \geq 0$. The system described by the RFDE (2) is 0–GAS if and only if there exist a locally Lipschitz functional $V : C([-\Delta, 0]; R^n) \rightarrow R^+$ and functions $\alpha_1$, $\alpha_2$ of class $K_\infty$, $\alpha_3$ of class $K$, such that, $\forall \phi \in C([-\Delta, 0]; R^n)$, the following inequalities hold:

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)$;

ii) $D^+ V(\phi, 0) \leq -a_3(|\phi(0)|)$
Theorem 8. (Pepe, Karafyllis & Jiang, 2006, 2008) The system described by the RFDE (2) is ISS if and only if there exist a locally Lipschitz functional $V : C([-\Delta, 0]; \mathbb{R}^n) \to \mathbb{R}^+$, a semi-norm $\| \cdot \|_a$ in $C([-\Delta, 0]; \mathbb{R}^n)$, functions $\alpha_1, \alpha_2$ of class $K_\infty$, functions $\alpha_3, \rho$ of class $K$ such that:

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a)$, $\forall \phi \in C$;

ii) $D^+V(\phi, d) \leq -\alpha_3(\|\phi\|_a),$

$\forall \phi \in C, d \in \mathbb{R}^m : \|\phi\|_a > \rho(|d|)$.

- Recall that $\gamma_a|\phi(0)| \leq \|\phi\|_a \leq \overline{\gamma}_a\|\phi\|_{\infty}$
Example of Copper Interconnections System for a Converter. More Red Regions Correspond to Higher Currents. Modelled by Partial Element Equivalent Circuits (PEECs).
Partial Element Equivalent Circuits (here an example is reported) describe electromagnetic problems, they are a circuit interpretation of the Maxwell Equations, when the space is suitably discretized. The electric and magnetic interactions do happen at distances and with propagation times, since the electromagnetic field propagates, at most, at the light speed. Thus delays are involved, which, in a state space description, affect both the state and its derivative (neutral-type systems). See papers by A. Bellen, N. Guglielmi, A. Ruehli, G. Antonini, X.-M. Zhang, Q.-L. Han, P. Pepe.
NFDEs in Hale’s Form

\[ \frac{d}{dt} (Dx_t) = f(x(t), x_t, v(t)), \quad t \geq 0, \text{ a.e.,} \]

\[ x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in C([-\Delta, 0]; \mathbb{R}^n) \quad (7) \]

where: \( x(t) \in \mathbb{R}^n; \) \( v(t) \in \mathbb{R}^m \) is the input, measurable and locally essentially bounded, \( n, m \) are positive integers; \( D : C([-\Delta, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^n \) is a map defined, for \( \phi \in C([-\Delta, 0]; \mathbb{R}^n), \) as

\[ D\phi = \phi(0) - q(\phi); \quad (8) \]

\( f, q \) Lipschitz on bounded sets.

So the equation (7) is read as follows

\[ \frac{d}{dt} (x(t) - q(x_t)) = f(x(t), x_t, v(t)), \quad t \geq 0, \text{ a.e.} \quad (9) \]
As well known (Rasvan, Niculescu) the neutral equation (7) can be rewritten, setting $\mathcal{D}x_t = \xi(t)$, as a couple of RFDE and Functional Difference Equation (FDE) as follows

$$
\dot{\xi}(t) = f(\xi(t) + q(x_t), x_t, v(t)), \text{ a.e.},
$$
$$
x(t) = \xi(t) + q(x_t),
$$
$$
x(\tau) = x_0(\tau), \ \tau \in [-\Delta, 0], \ \xi(0) = x_0(0) - q(x_0)
$$

(10)

Notice that $\xi(t)$ is (at least) continuous and the matching condition ($\xi(0) = x_0(0) - q(x_0)$) is clearly satisfied. The solution $x(t)$ is continuous (existence and uniqueness conditions will be given later).
A time invariant FDE is an equation of the type

\[
x(t) = g(x_t, u(t)), \quad t \geq 0,
\]

\[
x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in C([-\Delta, 0]; R^n),
\]

\(u\) continuous (corresponding to previous continuous variable \(\xi(t)\))

\(g\) Lipschitz on bounded sets, independent of the first argument at 0.

(11)

**Definition 9.** (see Hale & Lunel, 1993) A map \(g : C([-\Delta, 0]; R^n) \times R^m \rightarrow R^n\) is said to be independent of the first argument at 0 if there exists a real \(c \in (0, \Delta]\) such that, for any \(v \in R^m\) and for any \(\phi_1, \phi_2 \in C([-\Delta, 0]; R^n)\) satisfying \(\phi_1(\tau) = \phi_2(\tau), \tau \in [-\Delta, -c]\), the equality \(g(\phi_1, v) = g(\phi_2, v)\) holds.
By the independence assumption, the FDE is not implicit, the solution exists in $\mathbb{R}^+$. Assuming the matching condition $(x_0(0) = g(x_0, u(0)))$ (naturally satisfied by difference maps involved in NFDEs in Hale’s form), the solution is continuous.
Definition 10. Let in the FDE (11) \( u(t) = 0 \) \( \forall t \geq 0 \). The system described by the FDE (11) is said to be 0−GAS if there exists a function \( \beta \) of class \( \mathcal{KL} \) such that, for any \( x_0 \in C([-\Delta, 0]; R^n) \), the corresponding solution satisfies the inequality

\[
|x(t)| \leq \beta(\|x_0\|_{\infty}, t), \ \forall t \geq 0
\]  \hspace{1cm} (12)

Definition 11. The system described by the FDE (11) is said to be ISS, if there exist a function \( \beta \) of class \( \mathcal{KL} \) and a function \( \gamma \) of class \( \mathcal{K} \) such that, for any \( x_0 \in C([-\Delta, 0]; R^n) \) and any continuous input signal \( u \), satisfying the matching condition, the corresponding solution satisfies

\[
|x(t)| \leq \beta(\|x_0\|_{\infty}, t) + \gamma(\|u_{[0, t]}\|_{\infty}), \quad \forall t \geq 0
\]
Let, for any continuous function \( w : [0, c] \to \mathbb{R}^m \) and any \( \phi \in C([-\Delta, 0]; \mathbb{R}^n) \), satisfying the matching condition \( \phi(0) = g(\phi, w(0)) \), \( \phi_{c,w} \in C([-\Delta, 0]; \mathbb{R}^n) \) be defined, for \( s \in [-\Delta, 0] \), as

\[
\phi_{c,w}(s) = \begin{cases} 
\phi(s + c), & s \in [-\Delta, -c) \\
g(\phi^*_s, w(s + c)), & s \in [-c, 0],
\end{cases}
\]  
(13)

where \( \phi^*_s \in C([-\Delta, 0]; \mathbb{R}^n) \) is defined, for \( \theta \in [-\Delta, 0], s \in [-c, 0] \), as

\[
\phi^*_s(\theta) = \begin{cases} 
\phi(\theta + s + c), & \theta \in [-\Delta, -c - s) \\
\phi(0), & \theta \in [-c - s, 0],
\end{cases}
\]  
(14)
Theorem 12. The system described by the FDE (11), with \( u(t) = 0 \ \forall t \geq 0 \), is 0–GAS if and only if there exists a continuous functional \( V : C([−Δ, 0]; R^n) \rightarrow R^+ \), functions \( α_1, α_2 \) of class \( K_∞ \), a function \( α_3 \) of class \( K \), a semi-norm \( \| \cdot \|_a \) in \( C([−Δ, 0]; R^n) \) such that, \( ∀φ \in C([−Δ, 0]; R^n) : φ(0) = g(φ, 0) \), the inequalities hold:

i) \( α_1(\|φ(0)\|) \leq V(φ) \leq α_2(\|φ\|_a) \);

ii) \( V(φ_{c,0}) − V(φ) \leq −α_3(\|φ\|_a) \)

Recall that \( γ_a|φ(0)| \leq \|φ\|_a \leq γ_a\|φ\|_∞ \).
Theorem 13. Let there exists a continuous functional

\[ V : C([−\Delta, 0]; \mathbb{R}^n) \rightarrow \mathbb{R}^+, \]

functions \( \alpha_1, \alpha_2, \alpha_3 \) of class \( K_\infty \), a function \( \sigma \) of class \( K \), a semi-norm \( \| \cdot \|_a \) in \( C([−\Delta, 0]; \mathbb{R}^n) \) such that, for any \( \phi \in C([−\Delta, 0]; \mathbb{R}^n) \) and any continuous function \( w : [0, c] \rightarrow \mathbb{R}^m \), satisfying the matching condition \( \phi(0) = g(\phi, w(0)) \), the inequalities hold:

i) \( \alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a); \)

ii) \( V(\phi_{c,w}) − V(\phi) \leq −\alpha_3(\|\phi\|_a) + \sigma(\sup_{\tau \in [0,c]} |w(\tau)|) \)

Then, the system described by the FDE(11) is ISS.

Recall that \( \gamma_a|\phi(0)| \leq \|\phi\|_a \leq \overline{\gamma}_a\|\phi\|_\infty. \)
\[
\frac{d}{dt} (Dx_t) = f(x(t), x_t, v(t)), \quad t \geq 0, \text{ a.e.},
\]
\[x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in C([-\Delta, 0]; R^n),\]

(15)

where: \( x(t) \in R^n; v(t) \in R^m \) is the input, measurable and locally essentially bounded, \( n, m \) are positive integers; \( D : C([-\Delta, 0]; R^n) \rightarrow R^n \) is a map defined, for \( \phi \in C([-\Delta, 0]; R^n) \), as
\[
D\phi = \phi(0) - q(\phi) \quad (\text{for } x_t, \ D x_t = x(t) - q(x_t));
\]

(16)

\( q, f \) Lipschitz on bounded sets, respectively independent of the first argument and of the second argument at 0.
Lemma 14. The following results hold:

1) there exist, unique, a continuous solution \( x(t) \) of the NFDE in Hale’s form (15), on a maximal time interval \([0, b)\), \(0 < b \leq +\infty\);

2) the function \( t \to x(t) - q(x_t) \) is locally absolutely continuous in \([0, b)\);

3) if \( b < +\infty \), then the function \( t \to x(t) - q(x_t)\), \( t \in [0, b)\), is unbounded in \([0, b)\).
**Definition 15.** (see Hale & Lunel, 1993, Kolmanovskii & Myshkis, 1999, Khalil, 2000) The system described by the NFDE (15), with \( u(t) \equiv 0 \), is said to be 0-GAS if there exists a function \( \beta \) of class \( \mathcal{KL} \) such that, for any initial state \( \psi \in C([-\Delta, 0); \mathbb{R}^n) \), the solution exists for all \( t \geq 0 \) and, furthermore, it satisfies

\[
|x(t)| \leq \beta (\|\psi\|_{\infty}, t)
\]

(17)

**Definition 16.** (Sontag, 1989, Pepe, AUT, 2007) The system described by the NFDE (15) is said to be input-to-state stable if there exist a function \( \beta \) of class \( \mathcal{KL} \) and a function \( \gamma \) of class \( \mathcal{K} \) such that, for any initial state \( \psi \in C([-\Delta, 0]; \mathbb{R}^n) \) and any measurable, locally essentially bounded input \( v \), the solution exists for all \( t \geq 0 \) and, furthermore, it satisfies

\[
|x(t)| \leq \beta (\|\psi\|_{\infty}, t) + \gamma (\|v_{[0,t]}\|_{\infty})
\]

(18)
For a locally Lipschitz functional $V : C([−Δ, 0]; R^n) → R^+$, the derivative of the functional $V$, $D^+V : C([−Δ, 0]; R^n) × R^m → R^*$, is defined for $φ ∈ C([−Δ, 0]; R^n)$, $v ∈ R^m$, as

$$D^+V(φ, v) = \limsup_{h→0^+} \frac{1}{h} (V(φ_h,v) − V(φ)) \quad (19)$$

where: for $0 < h < Δ$, $φ_{h,v} ∈ C([−Δ, 0]; R^n)$ is given by

$$φ_{h,v}(s) = \begin{cases} 
φ(s + h), & s ∈ [−Δ, −h]; \\
Dφ + f(φ(0), φ, v)(s + h) − Dφ^*_s + h + φ(0), & s ∈ (−h, 0]; 
\end{cases} \quad (20)$$

for $0 < θ ≤ h$, $φ^*_θ ∈ C([−Δ, 0]; R^n)$ is given by

$$φ^*_θ(s) = \begin{cases} 
φ(s + θ), & s ∈ [−Δ, −θ]; \\
φ(0), & s ∈ (−θ, 0] 
\end{cases} \quad (21)$$
**Theorem 17.** Consider the NFDE (15). Let the system described by the FDE

\[
Dx_t = v(t), \quad t \geq 0,
\]

\[
x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in C([-\Delta, 0]; \mathbb{R}^n),
\]

be ISS with respect to the continuous input signal \( v(t) \). Let there exist a locally Lipschitz functional \( V : C([-\Delta, 0]; \mathbb{R}^n) \to \mathbb{R}^+ \), functions \( \alpha_1, \alpha_2 \) of class \( K_\infty \), and a function \( \alpha_3 \) of class \( K \) such that, \( \forall \phi \in C([-\Delta, 0]; \mathbb{R}^n) \), the inequalities hold:

\[
H_1 \quad \alpha_1 (|D\phi|) \leq V(\phi) \leq \alpha_2 (\|\phi\|_\infty);
\]

\[
H_2 \quad D^+V(\phi, 0) \leq -\alpha_3 (|D\phi|)
\]

(23)

Then, the system described by the NFDE (15) is 0-GAS.
If the operator $\mathcal{D}$ involved in (15) is linear and involves only discrete time-delays, i.e.

$$\frac{d}{dt} \left( x(t) - \sum_{k=1}^{p} A_k x(t - \Delta_k) \right) = f(x(t), x_t),$$

then the conditions provided in Theorem 17 are also necessary.

For the fully linear case (i.e., also the map $f$ is linear), constructive methodologies for finding the Lyapunov-Krasovskii functional are provided in V. Kharitonov, Time Delay Systems, Birkhauser, 2012.
**Theorem 18.** (Pepe & Karafyllis, IJC, 2013) Consider the NFDE (15). Let there exist a positive integer \( p \), \( p \) reals \( \Delta_i \in (0, \Delta], \) \( i = 1, 2, \ldots, p \) and \( p \) matrices \( A_i \in R^{n \times n}, i = 1, 2, \ldots, p \) such that

\[
\mathcal{D}\phi = \phi(0) - \sum_{k=1}^{p} A_k \phi(-\Delta_k) \tag{24}
\]

Let the system described by the FDE

\[
\mathcal{D}x_t = 0, \quad t \geq 0,
\]

\[
x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in C([-\Delta, 0]; R^n), \tag{25}
\]

be strongly stable (see Hale & Lunel, 1993). Then, the system described by the NFDE (15) is 0-GAS if and only if there exist a locally Lipschitz functional \( V : C([-\Delta, 0]; R^n) \to R^+ \), functions \( \alpha_1, \alpha_2 \) of class \( K_\infty \), and a function \( \alpha_3 \) of class \( K \) such that, \( \forall \phi \in C([-\Delta, 0]; R^n) \):

\[
H_1) \quad \alpha_1 (|\mathcal{D}\phi|) \leq V (\phi) \leq \alpha_2 (\|\phi\|_\infty)
\]

\[
H_2) \quad D^+V (\phi, 0) \leq -\alpha_3 (|\mathcal{D}\phi|) \tag{26}
\]
Theorem 19. (Pepe, Karafyllis & Jiang, AUT, 2008) Consider the NFDE (15). Let the system described by the FDE

\[ \dot{x}_t = v(t), \quad t \geq 0, \]
\[ x(\tau) = x_0(\tau), \quad \tau \in [-\Delta, 0], \quad x_0 \in C([-\Delta, 0]; R^n), \]

be ISS with respect to the continuous input signal \( v(t) \). Let there exist a locally Lipschitz functional \( V : C([-\Delta, 0]; R^n) \rightarrow R^+ \), functions \( \alpha_1, \alpha_2 \) of class \( K_\infty \), and functions \( \alpha_3, \rho \) of class \( K \) such that:

\[ H_1) \ \alpha_1(|D\phi|) \leq V(\phi) \leq \alpha_2 \left( \left\| \begin{bmatrix} D\phi \\ \phi \end{bmatrix} \right\|_a \right), \quad \forall \ \phi \in C([-\Delta, 0]; R^n); \]
\[ H_2) \ \dot{D}^+ V(\phi, u) \leq -\alpha_3 \left( \left\| \begin{bmatrix} D\phi \\ \phi \end{bmatrix} \right\|_a \right), \]

\[ \forall \ \phi \in C([-\Delta, 0]; R^n), \ u \in R^m : \left\| \begin{bmatrix} D\phi \\ \phi \end{bmatrix} \right\|_a > \rho(|u|) \]  \hspace{1cm} (28)

Then, the system described by the NFDE (15) is ISS.
\[ \dot{x}(t) = f(x_t) + g(x_t)v(t), \quad t \geq 0, \quad a.e., \]
\[ x(\tau) = \xi_0(\tau), \quad \tau \in [-\Delta, 0], \quad (29) \]

\[ x_t \in C([-\Delta, 0]; \mathbb{R}^n), \quad x_t(\tau) = x(t + \tau) \]
ISS-ation w.r.t. the Actuator Disturbance

\[
\dot{x}(t) = f(x_t) + g(x_t)(u(t) + d(t)), \quad t \geq 0, \quad a.e.,
\]
\[
x(\tau) = \xi_0(\tau), \quad \tau \in [-\Delta, 0], \quad (30)
\]

\(x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m\) control input, \(d(t) \in \mathbb{R}^m\) unknown disturbance, supposed Lebesgue measurable and locally essentially bounded.

PROBLEM: given a state feedback \(k(x_t)\) such that
\[
\dot{x}(t) = f(x_t) + g(x_t)k(x_t)
\]
is 0-GAS, find a new state feedback \(k(x_t) + p(x_t)\) such that
\[
\dot{x}(t) = f(x_t) + g(x_t)(k(x_t) + p(x_t) + d(t))
\]
is ISS w.r.t. \(d(t)\).
For given $\phi \in C([−\Delta, 0]; R^n)$, $h \in [0, \Delta)$, let $\phi^g_h \in C([−\Delta, 0]; R^{n \times m})$ be defined as

$$\phi^g_h(s) = \begin{cases} 
0 & s \in [−\Delta, −h) \\
(s + h)g(\phi) & s \in [−h, 0]
\end{cases}$$

(31)
Theorem for ISS-ation, Pepe, TAC, 2009

Hp) There exist a Lipschitz on bounded sets functional

\[ k : C([-\Delta, 0]; \mathbb{R}^n) \to \mathbb{R}^m, \]

a continuously Fréchet differentiable functional

\[ V : C([-\Delta, 0]; \mathbb{R}^n) \to \mathbb{R}^+, \]

functions \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) of class \( \mathcal{K}_\infty \), such that, along the solutions of the unforced (disturbance equal to zero) closed loop system (30) with \( u(t) = k(x_t) \), described by

\[
\dot{x}(t) = f(x_t) + g(x_t)k(x_t), \tag{32}
\]

the following inequalities hold:

i) \( \alpha_1(||\phi(0)||) \leq V(\phi) \leq \alpha_2(||\phi||_a) \);

ii) \( D^+V(\phi) \leq -\alpha_3(||\phi||_a) \)
The feedback control law

\[ u(t) = k(x_t) + p(x_t), \quad (33) \]

with

\[ p = \begin{bmatrix} p_1 & p_2 & \ldots & p_m \end{bmatrix}^T : C([-\Delta, 0]; \mathbb{R}^n) \to \mathbb{R}^m \]

declared as

\[ p_i(\phi) = -\limsup_{h \to 0^+} D_F V(\phi) \frac{1}{h} \phi_h^g e_i, \quad (34) \]

\( e_i \) being the canonical basis in \( \mathbb{R}^m \), is such that the closed loop system (30), (33), described by

\[ \dot{x}(t) = f(x_t) + g(x_t)k(x_t) + g(x_t)p(x_t) + g(x_t)d(t), \quad (35) \]

is input-to-state stable with respect to the measurable and locally essentially bounded disturbance \( d(t) \), provided that the functional \( p \) is Lipschitz on bounded sets of \( C([-\Delta, 0]; \mathbb{R}^n) \).
briefly...

\[ \dot{x}(t) = f(x_t) + g(x_t)(u(t) + d(t)), \quad (36) \]

Hyp) \( u(t) = k(x_t) \) is stabilizing in the unforced case \((d(t) = 0)\), \( V \) is a L-K functional for \( \dot{x}(t) = f(x_t) + g(x_t)k(x_t) \)

Th) For \( p_i(\phi) = \limsup_{h \to 0^+} D_F V(\phi) \frac{1}{h} \phi_h^g e_i \),

\[ u(t) = k(x_t) + p(x_t) \]

is input-to-state stabilizing, i.e

\[ \dot{x}(t) = f(x_t) + g(x_t)(k(x_t) + p(x_t) + d(t)) \]

is ISS w.r.t. \( d(t) \).
Disturbance Attenuation

\[ |x(t)| \leq \beta(\|\xi_0\|_{\infty}, t) + \gamma(\|d_{[0,t]}\|_{\infty}) \]

\[ \gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \left( \frac{s^2}{3} \right) \]

If, instead of \( V \), we choose \( \omega V \), with \( \omega \) a positive real, then

\[ \gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1} \left( \frac{s^2}{3 \omega} \right) \]

The disturbance can be arbitrarily attenuated. Price to pay: \( p(x_t) \) becomes \( \omega p(x_t) \) (i.e., increased control effort).
Further extensions to the case of saturated input and equations with discontinuous right-hand side, can be found in Pepe & Ito, TAC 2012. Invariantly differentiable functionals (Kim, 1997) are used.

Small-gain theory for ISS and integral ISS (iISS) of interconnected systems can be found in Karafyllis & Jiang, SIAM, 2007, Ito, Pepe & Jiang, AUT, 2010.
Scheme of a Continuous Stirred Tank Reactor
\[
\frac{dC_A(t)}{dt} = \frac{F}{V_R} \left( \phi C_{A0} + (1 - \phi) C_A(t - \Delta) - C_A(t) \right) - C_A(t) k_0 e^{-E R T_R(t)}
\]

\[
\frac{dT_R(t)}{dt} = \frac{F}{V_R} \left( \phi T_0 + (1 - \phi) T_R(t - \Delta) - T_R(t) \right) - \frac{\lambda C_A(t) k_0 e^{-E R T_R(t)}}{\rho c_p} - \frac{U A_J (T_R(t) - T_J(t))}{V_R \rho c_p}
\]

\[
\frac{dT_J(t)}{dt} = \frac{F_J(t)}{V_J} \left( T_{C,in} - T_J(t) \right) + \frac{V_R \rho c_p}{U A_J (T_R(t) - T_J(t))}
\]

\[
F_J(t) = u(t) + d(t)
\]
In the case the disturbance is not present \((d(t) \equiv 0)\), a stabilizing feedback control law

\[
u(t) = k((T_R)_t, (C_A)_t, (T_J)_t)\]

is found by tools of differential geometry for time-delay systems (Germani, Manes, Pepe, Oguchi, Watanabe, Nakamizo, Marquez-Martinez, Moog). The closed-loop system (with \(u = k\)) becomes

\[
\dot{E}(t) = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} N(E_1(t)) + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{F}{V_R}(1 - \Phi)
\end{bmatrix} E(t - \Delta) \\
A_B + B_B K \\
0 & 0 & -\frac{F}{V_R} - k_0 e^{-E R \left( E_1(t) + T_{R,eq} \right)} \\
UA_J (T_{C,in} - \mathcal{F}(E(t), E(t-\Delta))) \\
\frac{0}{V_J V_R \rho c_p} \\
0
\end{bmatrix} E(t) + \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} d(t)
\]

\(38\)
A functional $V$ by which the asymptotic stability of the unforced ($d(t) \equiv 0$) closed loop system can be proved, its Fréchet Differential and the related ISS-ing term $p$ in the control law are the following ($\phi, \psi, \phi_h^g \in C([-\Delta, 0]; R^3)$, (Pepe & Di Ciccio, IJRNC 2011)

$$V(\phi) = \phi^T(0)P\phi(0) + \int_{-\Delta}^{0} \phi^T(\tau) \left( -\frac{\tau}{\Delta}Q_1 + \frac{\tau + \Delta}{\Delta}Q_2 \right) \phi(\tau) d\tau$$

$$D_FV(\phi)\psi = 2\phi^T(0)P\psi(0) + 2 \int_{-\Delta}^{0} \phi^T(\tau) \left( -\frac{\tau}{\Delta}Q_1 + \frac{\tau + \Delta}{\Delta}Q_2 \right) \psi(\tau) d\tau$$

$$D_FV(\phi)\frac{1}{h}\phi_h^g = \frac{1}{h}2\phi^T(0)Phg(\phi)$$

$$+ \frac{1}{h}2 \int_{-h}^{0} \phi^T(\tau) \left( -\frac{\tau}{\Delta}Q_1 + \frac{\tau + \Delta}{\Delta}Q_2 \right) (\tau + h)g(\phi) d\tau$$

$$p(\phi) = -2\phi^T(0)Pg(\phi)$$
\[
p((T_R)_t, (C_A)_t, (T_J)_t) =
\begin{pmatrix}
T_R(t) - T_{R,eq} \\
-2 \frac{F}{V_R} \left( \phi T_0 + (1 - \phi) T_R(t - \Delta) - T_R(t) \right) \\
- \frac{\lambda C_A(t) k_0 e^{RT_R(t)}}{\rho c_p} - \frac{U A_J (T_R(t) - T_J(t))}{V_R \rho c_p} \\
C_A(t) - C_{A,eq}
\end{pmatrix}^T
\]

\[
T
\begin{pmatrix}
0 \\
UA_J(T_{C, in} - T_J(t)) \\
\frac{V_J V_R \rho c_p}{V_R \rho c_p}
\end{pmatrix}
\]

(39)
\[ u(t) = k((T_R)_t, (C_A)_t, (T_J)_t) \]

stabilizes (locally) the unforced \( (d(t) \equiv 0) \) system.

\[ u(t) = k((T_R)_t, (C_A)_t, (T_J)_t) + p((T_R)_t, (C_A)_t, (T_J)_t) \]

input-to-state stabilizes locally the system with respect to the unknown disturbance \( d(t) \) adding to the control input, with significant disturbance effect attenuation.

In the following simulations

\[ d(t) = 0.2F_{J,eq} + 0.4F_{J,eq} \cos(0.001t) \]
Reactor Temperature, $u = k$
Reactor Temperature, $u = k + p$
Control Signal, \( u = k + p \)
**Conclusions**

- Liapunov-Krasovskii Characterizations of ISS for systems described by RFDEs, FDEs, NFDEs have been presented.

- Formulas for the input-to-state stabilization of retarded nonlinear systems are provided, by means of Fréchet differentiable functionals.

- Such formulas extend the ones given by Sontag in 1989 for delay-free nonlinear systems.

- This theoretical result has been applied to the model of a continuous stirred tank reactor, showing the high performance of the new projected control law, as far as the attenuation of the actuator disturbance effect is concerned.
Future Developments

Main Future applications of the ISS theory may concern the use of Control Lyapunov-Krasovskii functionals for the stabilization, the practical stabilization, the input-to-state practical stabilization of systems described by nonlinear RFDEs, FDEs, NFDEs.

As far as the stabilization of RFDEs is concerned, the (Control) Lyapunov-Razumikhin methodology has been investigated in Jankovic, TAC, 2001, and the (Control) Lyapunov-Krasovskii methodology has been investigated in Karafyllis & Jiang, Springer, 2011.

Further important developments concern the ISS and the integral ISS of networked systems described by RFDEs, NFDEs.
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